

DISS. ETH NO. 22118

PhD Thesis

**A Hardy Space Approach to
Lagrangian Floer gluing**

Tatjana SIMČEVIĆ

ETH Zurich

October 2014

Acknowledgement

First and foremost I would like to express my deep gratitude to my advisor, Professor Dietmar Salamon, for sharing his enthusiasm and insight into the subject, as well as for his guidance and help during the whole doctorate.

I would like to thank the co-examiners Professor Paul Biran and Professor Urs Frauenfelder for their interest in my work.

I thank to my colleagues from the ETH for the enjoyable working atmosphere and pleasant moments spent together. This work was partially supported by the Swiss National Science Foundation Grant 200021-127136 and I would like to thank them for their financial support.

Above all I would like to thank to my parents Velibor and Stojka, my brother Dalibor and my boyfriend Saša for their encouragement, support and understanding.

Abstract

We develop a new approach to Lagrangian-Floer gluing. The construction of the gluing map is based on the intersection theory in some Hilbert manifold of paths \mathcal{P} . We consider some moduli spaces of perturbed holomorphic curves whose domains are either strips or more general Riemann surfaces with strip-like ends. These moduli spaces can be injectively immersed, by taking the restriction to non-Lagrangian boundary, into the Hilbert manifold \mathcal{P} .

Then we restrict our attention to some subsets $\mathcal{M}^\infty(\mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$ of the aforementioned moduli spaces of perturbed holomorphic strips. These moduli spaces consist of small energy curves whose non-Lagrangian boundary is contained in the neighbourhood \mathcal{U} of a fixed Hamiltonian path x . Monotonicity results will guarantee that the elements of these moduli spaces are contained in some neighbourhood of the Hamiltonian path x . This will allow us to carry over the main analysis in suitable local coordinate charts.

The moduli spaces $\mathcal{M}^T(\mathcal{U})$ and $\mathcal{M}^\infty(\mathcal{U})$ turn out to be embedded submanifolds of the Hilbert manifold of paths \mathcal{P} . We prove that $\mathcal{M}^T(\mathcal{U})$ converges in the C^1 topology toward $\mathcal{M}^\infty(\mathcal{U})$. As an application of this convergence result we prove various gluing theorems. We explain the construction of Lagrangian-Floer homology and prove that the square of the boundary map is equal to zero. Here we restrict our discussion to the monotone case with minimal Maslov number at least three. We also prove that the homology is independent of the Hamiltonian and almost complex structure used in its definition. We include the exposition of the Lagrangian-Floer-Donaldson functor and Seidel homomorphism.

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Chapter 1

Introduction

Floer homology was developed in 1980's by A. Floer in the series of papers [4, 5, 6, 7] and today it has various application in low dimensional topology and symplectic and contact topology. Three of its main technical ingredients are Gromov-compactness, transversality-Fredholm theory and gluing. In this thesis we shall not discuss compactness and transversality, but we shall develop a new approach to gluing.

When we say gluing, we always think of it as the opposite of Gromov-compactness. Particularly, this means that two holomorphic curves (strips), that intersect in the appropriate sense can be “glued” together if they can be approximated by some nearby genuine holomorphic curve. Floer's idea was first to use cut-off functions and to construct a so called pregluing curve, which in general doesn't have to be holomorphic and then using the implicit function theorem to establish the existence of an actual nearby holomorphic curve. Our approach is based on the intersection theory in Hilbert manifolds. More precisely, we consider two different moduli spaces of pairs of holomorphic curves whose domains are either half-infinite strips or more general truncated Riemann surfaces with strip-like ends. These Hilbert manifolds are infinite dimensional and they can be injectively immersed into some Hilbert manifold of paths \mathcal{P} . The images of these moduli spaces represent Hardy submanifolds of \mathcal{P} , mentioned in the title. The existence of a nearby holomorphic curve is provided by the existence of a unique intersection point of these two submanifolds. This approach is discussed in more details below.

In chapter 5 we explain how the Hardy space gluing theory developed in this thesis is used in the construction of Floer homology. In this chapter the exposition is restricted to monotone Lagrangian submanifold with minimal Maslov number at least three. The rest of the thesis doesn't assume any restrictions on the Lagrangian submanifolds and could potentially be used in more general cases. In [17] besides the same assumption on the minimal

Maslov number was also assumed that the image $\text{Im}(\pi_1(L_i)) \subset \pi_1(M)$ is a torsion subgroup, what allowed them to work with \mathbb{Z}_2 coefficients. We will not assume this condition and we don't impose any restrictions on the monotonicity factors, which forces us to work with Novikov rings. For the reader who is not familiar with Lagrangian Floer homology we shortly describe its construction and then we describe how the technical part of our new approach to gluing is applied in the relevant construction.

The Floer chain complex is a vector space over some Novikov ring Λ generated by Hamiltonian paths. The boundary operator ∂ is defined by counting (mod 2) the number of elements of some zero-dimensional moduli space. This moduli space consists of perturbed holomorphic strips that connect two Hamiltonian paths. As the first application of our gluing approach we prove that the square of the boundary map is equal zero. This allows us to define Lagrangian Floer homology $HF(L_0, L_1; H, J)$.

The proof of the identity $\partial^2 = 0$ is based on the study of the 2-dimensional moduli space of index two perturbed holomorphic strips that connect two Hamiltonian paths x and z . This moduli space $\mathcal{M}^2(x, z; H, J)$ allows free \mathbb{R} action by translation. The quotient space $\widehat{\mathcal{M}}^2 = \mathcal{M}^2(x, z; H, J)/\mathbb{R}$ will in general not be compact, but it can be compactified by adding the zero dimensional product space consisting of broken trajectories. The aim of the gluing theorem is to identify an end of $\widehat{\mathcal{M}}^2$ with a broken trajectory. More precisely, the purpose of the Floer gluing theorem is to construct a diffeomorphism of an interval $(T_0, +\infty)$ and some open subset of $\widehat{\mathcal{M}}^2$, such that compactification of the interval, i.e. adding infinity, corresponds to adding the broken trajectory. As a broken trajectory we mean a pair (u, v) , where u is a trajectory connecting x and y , whereas v connects y and z . Gluing map assigns to each $T \in (T_0, \infty)$ a holomorphic curve $u_T \in \mathcal{M}^2(x, z; H, J)$ such that after shifting by T on the left u_T converges to u , and on the right to v . Construction of the curve u_T is the moment when our new approach provides an alternative method.

The curve u_T is obtained as the isolated intersection point of two Hilbert manifolds $\mathcal{M}^T(y, \mathcal{U})$ and $\mathcal{M}^-(x) \times \mathcal{M}^+(z)$. The manifold $\mathcal{M}^T(y, \mathcal{U})$ consists of perturbed holomorphic curves with small energy whose domain is a strip $[-T, T] \times [0, 1]$ and which are the boundary $\{\pm T\} \times [0, 1]$ contained in the neighborhood \mathcal{U} of a Hamiltonian path y , whereas the other two boundary components $[-T, T] \times \{0, 1\}$ lie on Lagrangian submanifolds. The other moduli space $\mathcal{M}^-(x) \times \mathcal{M}^+(z)$ consists of pairs of half infinite strips with fixed energy converging to x and z respectively. We prove that these moduli spaces are indeed infinite dimensional submanifolds of some Hilbert manifold of $W^{2,2}$ strips. We also consider another Hilbert manifold $\mathcal{M}^\infty(y, \mathcal{U})$ consist-

ing of pairs of half infinite holomorphic curves which have sufficiently small energy and are at the boundary contained in \mathcal{U} . The bound on the energy and the neighborhood of $y - \mathcal{U}$ are chosen in such a way that the elements of $\mathcal{M}^\infty(y, \mathcal{U})$ as well as $\mathcal{M}^T(y, \mathcal{U})$ are confined to a small neighborhood of y . In chapter 2 we prove some monotonicity results which guarantee that the non-Lagrangian (free) boundary and the energy of a perturbed holomorphic curve control its diameter. Thus, these monotonicity results imply that the elements of these moduli spaces are contained in a small neighborhood of a Hamiltonian path y and therefore the main analysis can be carried out in Euclidean space using appropriate coordinate charts.

We also prove that the manifolds $\mathcal{M}^T(y, \mathcal{U})$ and $\mathcal{M}^\infty(y, \mathcal{U})$ can be embedded by taking the restriction to the boundary into some Hilbert manifold of paths $\mathcal{P} \times \mathcal{P}$. Their images \mathcal{W}^T and \mathcal{W}^∞ are the nonlinear Hardy spaces of the title, they are exactly those paths that can be extended holomorphically to the corresponding strips. One of the most important result of the thesis is that \mathcal{W}^T converge to \mathcal{W}^∞ in C^1 topology. If we now go back to the construction of the map u_T in the neighborhood of the broken trajectory (u, v) , we notice that a pair of paths $(u(0), v(0)) \in \mathcal{P} \times \mathcal{P}$ is an isolated intersection point of \mathcal{W}^∞ and $\mathcal{M}^-(x) \times \mathcal{M}^+(z)$. As \mathcal{W}^T converge towards \mathcal{W}^∞ this implies that for T sufficiently large there will be a unique isolated intersection point of \mathcal{W}^T and $\mathcal{M}^-(x) \times \mathcal{M}^+(z)$, which we denote by u_T .

Another application of our gluing approach is in the construction of the Floer connection homomorphism. Namely, for two regular pairs (H^α, J^α) and (H^β, J^β) we can consider homotopy connecting these two pairs and the corresponding time dependent Floer equation. The mod two count of solutions of such an equation defines homomorphism of the corresponding chain complexes $\Phi^{\beta\alpha}$. We prove that this homomorphism intertwines the Floer boundary operator and hence defines a homomorphism of the corresponding homology groups. The proof is again based on the study of some 1-dimensional moduli space and the identification of its ends with the images of some gluing maps.

The homomorphisms $\Phi^{\beta\alpha}$ are independent of the choice of the homotopy connecting the regular pairs (H^α, J^α) and (H^β, J^β) . Thus two homomorphisms $\Phi_1^{\beta\alpha}$ and $\Phi_0^{\beta\alpha}$ defined using two different homotopies are chain homotopy equivalent, and hence induce the same map on the homology level.

Finally we prove that two composable morphisms satisfy the composition rule under the catenation of homotopies. More precisely three regular pairs, (H^α, J^α) , (H^β, J^β) and (H^γ, J^γ) and the corresponding homomorphisms $\Phi^{\beta\alpha}$ and $\Phi^{\gamma\beta}$ satisfy the composition rule $\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} = \Phi^{\gamma\alpha}$ and $\Phi^{\alpha\alpha} = \text{Id}$. Thus, this implies that the Lagrangian Floer homology doesn't depend on the

choice of the Hamiltonian and the almost complex structure. This is where the presence of Novikov ring makes the analysis much more difficult and the proof requires the construction of infinitely many morphisms $\Phi_\nu^{\gamma\alpha}$ such that the composition $\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha}$ corresponds to the limit $\lim_{\nu \rightarrow \infty} \Phi_\nu^{\gamma\alpha}$ and $\Phi^{\gamma\alpha} = \Phi_0^{\gamma\alpha}$. Each two consecutive morphisms $\Phi_\nu^{\gamma\alpha}$ are chain homotopic equivalent.

The above properties of the homomorphisms $\Phi^{\beta\alpha}$ and the corresponding gluing theorems are done not just in the case of strips, but also in the case of more general Riemann surfaces with finitely many half-infinite strip-like ends. These results give rise to a Lagrangian Floer-Donaldson functor from the category of Lagrangian pairs to the category of vector spaces over the Novikov ring Λ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. The objects are finite tuples of pairs of monotone Lagrangian submanifolds and the morphisms, called *string cobordisms*, are 2-manifolds with boundary and finitely many strip-like ends and a monotone Lagrangian submanifold for each boundary component. The Lagrangian Floer-Donaldson functor assigns to each tuple of Lagrangian pairs the tensor product of its Floer homology groups and to each string cobordism a morphism on Floer homology. The last chapter of the thesis also includes the exposition of the Lagrangian Seidel homomorphism.

The thesis is organized as follows. In the second chapter we prove the monotonicity lemma for holomorphic curves with Lagrangian boundary conditions. As a corollary we prove some results which guarantee that holomorphic strips with small energy and some condition on the non-Lagrangian boundary are localized near Lagrangian intersection point and hence can be studied in suitable local coordinates. We also prove exponential decay of finite energy holomorphic strips with Lagrangian boundaries in the case of tame almost complex structure and clean intersection of Lagrangian submanifolds.

In the third chapter we prove some linear elliptic estimates which will be crucial for the proofs of the main theorems in the fourth chapter. We also prove elliptic regularity and surjectivity of some specific linearized operator whose domain represent $W^{2,2}$ maps on strips. Additional difficulties that don't occur in standard Floer theory appear because of the presence of truncated surfaces, i.e. domains with corners. The corresponding linearized operator will not be Fredholm in this case, but it will still be surjective. In the appendix of this chapter we discuss the abstract interpolation theory, as it has various applications through the whole thesis.

The fourth chapter contains the exposition of the Hilbert manifold setup necessary for the statement of the main theorems. In the appendix of this chapter we discuss further the interpolation Lions-Magenes [15] spaces which represent the model space of the aforementioned Hilbert manifold of paths \mathcal{P} . This chapter contains the statement and the proofs of the main theo-

rems. We prove that \mathcal{W}^T converges in C^1 topology to \mathcal{W}^∞ and we prove that they are embedded submanifolds of the Hilbert path space $\mathcal{P} \times \mathcal{P}$. We also prove that $\mathcal{M}^\pm(x)$, consisting of perturbed holomorphic strips that converge to x , is a Hilbert submanifold of some Hilbert manifold of strips \mathcal{B} and that it can be injectively immersed into the path manifold \mathcal{P} .

The last chapter is joint work with Prof. D. Salamon and it contains various applications of the results from the third chapter. We explain how the Floer gluing theorems can be reduced to intersection theory in the path space \mathcal{P} . We prove that the square of the boundary map is equal zero and we prove the aforementioned properties of the homomorphism $\Phi^{\beta\alpha}$. We include the exposition of the Floer-Donaldson category.

Chapter 2

Monotonicity for holomorphic curves

2.1 Main results

It is well known that the monotonicity lemma holds for minimal surfaces $u : \Omega \rightarrow M$. *Monotonicity* means that the area of the piece of the surface $u(\Omega)$, cut from $u(\Omega)$ by a small ball of radius r and centered on the surface, is bounded below by $c \cdot r^2$. It is well known that this holds for holomorphic curves in the case that the boundary $u(\partial\Omega)$ isn't contained in the ball $B_r(u(z))$. We shall prove that monotonicity holds also if

$$u(\partial\Omega) \cap B_r(u(z_0)) \neq \emptyset$$

It suffices that the piece of a boundary within the ball lies on Lagrangian submanifolds which intersect cleanly. We illustrate this phenomenon in Figure 2.1.

Throughout this chapter we shall assume the following:

- (H) (M, ω) is a symplectic manifold without boundary and $L_0, L_1 \subset M$ are Lagrangian submanifolds without boundary that are closed subsets of M . The intersection $\Lambda = L_0 \cap L_1$ is compact and $N \subset M$ is a compact neighborhood of Λ .

The main theorem of this chapter is the following monotonicity theorem for holomorphic curves with Lagrangian boundary conditions.

Theorem 2.1.1 (Monotonicity lemma). *Assume (H) and let J be an ω -tame almost complex structure. Suppose that the intersection $\Lambda = L_0 \cap L_1$*

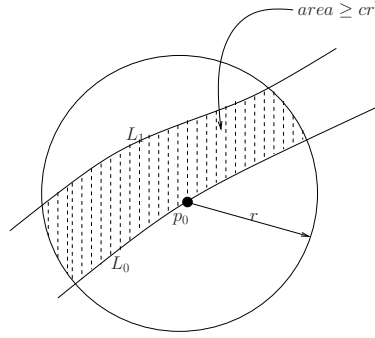


Figure 2.1: Monotonicity for curves with Lagrangian boundary

is clean. Equip M with the metric $\langle \xi, \eta \rangle = \frac{1}{2}(\omega(\xi, J\eta) + \omega(\eta, J\xi))$. There exist positive constants c_0 and r_0 such that the following holds. Let $\Omega \subset \mathbb{R} \times [0, 1]$ be a bounded open set, $(s_0, t_0) \in \Omega$, $0 < r \leq r_0$. If a J -holomorphic curve $u : \Omega \rightarrow N$ which extends continuously to $\overline{\Omega}$ satisfies the following

$$\begin{aligned} u(s, 0) &\in L_0, \text{ for all } s \in \mathbb{R} \text{ with } (s, 0) \in \Omega, \\ u(s, 1) &\in L_1, \text{ for all } s \in \mathbb{R} \text{ with } (s, 1) \in \Omega \end{aligned} \quad (2.1)$$

and

$$u(\overline{\Omega} \setminus \Omega) \cap \overline{B_r(p_0)} = \emptyset, \quad (2.2)$$

where $p_0 = u(s_0, t_0)$. Then

$$\int_{u^{-1}(B_r(p_0))} u^* \omega \geq c_0 r^2. \quad (2.3)$$

Proof. See Section 2.4. □

The following standard monotonicity lemma (see for example [11]) for holomorphic curves without boundary is a corollary of Theorem 2.1.1.

Corollary 2.1.2. *Let (M, ω) be a symplectic manifold and $N \subset M$ compact. Let J be an ω -tame almost complex structure and equip M with the metric $g = \frac{1}{2}(\omega(\cdot, J\cdot) + \omega(J\cdot, \cdot))$. Then there are constants $\epsilon_0, c_0 > 0$ such that the following holds. If $\Omega \subset \mathbb{C}$ is an open and bounded set, $(s_0, t_0) \in \Omega$, $0 < r < \epsilon_0$ and*

- $u : \Omega \rightarrow N$ is a smooth J -holomorphic curve.
- $u : \overline{\Omega} \rightarrow N$ is continuous.
- $u(\overline{\Omega} \setminus \Omega) \cap \overline{B_r(u(s_0, t_0))} = \emptyset$ for $r \leq \epsilon_0$

Then

$$\int_{u^{-1}(B_r(u(s_0, t_0)))} u^* \omega \geq c_0 r^2 \quad (2.4)$$

Proof. After rescaling we may assume $\Omega \subset \mathbb{R} \times [0, 1]$. Then the boundary conditions are vacuous, so the assertion follows from Theorem 2.1.1. \square

Extension to the case of t -dependent almost complex structures

Let J_t be a smooth family of ω -tame almost complex structures and let $I \subset \mathbb{R}$ be an interval. We consider J_t -holomorphic curves with Lagrangian boundary conditions, i.e. solutions $u : I \times [0, 1] \rightarrow N \subset M$ of the following boundary value problem

$$\partial_s u + J_t(u) \partial_t u = 0, \quad u(s, i) \in L_i, i = 0, 1. \quad (2.5)$$

Such holomorphic curves were studied by Floer [4, 5, 6] and were used to define the Floer homology of Lagrangian intersection. The need of introducing t -dependent almost complex structure comes from transversality issues. See for example [8], where the construction leading to transversality involved such almost complex structures. Denote with $E(u)$ the energy of such a holomorphic curve

$$E(u) = \int_{I \times [0, 1]} u^* \omega = \int_{I \times [0, 1]} |\partial_s u|_{J_t}^2 ds dt. \quad (2.6)$$

A set $\Lambda_0 \subset L_0 \cap L_1$ is called an **isolated set** of intersections if there is an open neighborhood $V \subset M$ of Λ_0 with compact closure such that $\overline{V} \cap L_0 \cap L_1 = \Lambda_0$. In particular Λ_0 is compact. Any such V is called an **isolating neighborhood** of Λ_0 . Our second main result states that the energy of u and the non-Lagrangian boundary $u|_{\partial I \times [0, 1]}$ control the diameter of u .

Theorem 2.1.3. *Assume (H). Suppose that $\Lambda_0 \subset L_0 \cap L_1$ is an isolated set of intersections. Let U be an open neighborhood of Λ_0 and let V be an isolated neighborhood of Λ_0 such that $\overline{V} \subset U$. There exists \hbar such that the following holds. For any interval $I = [a, b] \subset \mathbb{R}$ if a solution $u : I \times [0, 1] \rightarrow N \subset M$ of (2.5), satisfies*

$$E(u) < \hbar, \quad \text{and} \quad u|_{\partial I \times [0, 1]} \in V,$$

then

$$u(s, t) \in U, \quad \forall (s, t) \in I \times [0, 1].$$

Proof. See Section 2.4. □

As a corollary of Theorem 2.1.3 we prove an analogous result for perturbed holomorphic strips. Let

$$[0, 1] \times M \rightarrow \mathbb{R} : (t, p) \mapsto H(t, p) = H_t(p)$$

be a time dependent Hamiltonian function. Denote by $\phi_t : \Omega_t \rightarrow M$ the Hamiltonian isotopy generated by H via

$$\partial_t \phi_t(p) = X_{H_t}(\phi_t(x)), \quad \phi_0 = \text{Id}$$

Thus $\Omega_t \subset M$ is the open set of all points $p_0 \in M$ such that the solution of the initial value problem $\dot{x}(s) = X_{H_s}(x(s))$, $x(0) = p_0$, exists on the interval $[0, t]$.

Theorem 2.1.4. *Assume (H) and let $J = \{J_t\}_{0 \leq t \leq 1} \in \mathcal{J}(M, \omega)$. Suppose $\Lambda \subset L_0 \cap \phi_1^{-1}(L_1)$ is an isolated set of intersections, let $U \subset M$ be an open neighborhood of Λ , and let $V \subset M$ be an isolating neighborhood of Λ such that $\bar{V} \subset U$. Fix a compact set $K \subset M$. There is a constant $\hbar > 0$ such that the following holds. If $T > 0$ and $u : [-T, T] \times [0, 1] \rightarrow K$ satisfies*

$$\partial_s u + J_t(u) (\partial_t u - X_{H_t}(u)) = 0, \quad u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \quad (2.7)$$

and

$$E_H(u) = \int_{-T}^T \int_0^1 |\partial_s u|^2 dt ds < \hbar, \quad u(\{\pm T\}, t) \subset \phi_t(V) \quad \forall t \in [0, 1], \quad (2.8)$$

then $u(s, t) \in \phi_t(U \cap \Omega_t)$ for all $s \in [-T, T]$ and all $t \in [0, 1]$.

Proof. This theorem was proved in 2.1.3 for $H = 0$. We reduce the general case to the case $H = 0$ in two steps.

Step 1. *It suffices to assume that the Hamiltonian $H : [0, 1] \times M \rightarrow \mathbb{R}$ has compact support (and hence $\Omega_t = M$ for $0 \leq t \leq 1$).*

Shrinking U , if necessary, we may assume that \bar{U} is compact and

$$\bar{V} \subset U \subset \bar{U} \subset \Omega_1.$$

Then the set

$$\hat{K} := [0, 1] \times K \cup \bigcup_{0 \leq t \leq 1} \{t\} \times \phi_t(\bar{U})$$

is a compact subset of $[0, 1] \times M$. Replace the Hamiltonian function H by a function $\hat{H} : [0, 1] \times M \rightarrow \mathbb{R}$ with compact support that agrees with H on a

neighborhood of \widehat{K} . Denote by $\widehat{\phi}_t$ the Hamiltonian isotopy generated by \widehat{H} . Then $\widehat{\phi}_t(U) = \phi_t(U \cap \Omega_t)$ and $\widehat{\phi}_t(V) = \phi_t(V)$ for $0 \leq t \leq 1$. Hence a function $u : [-T, T] \times [0, 1] \rightarrow K$ satisfies (2.7) and (2.8) if and only if it satisfies the same conditions with H and ϕ_t replaced by \widehat{H} and $\widehat{\phi}_t$. This proves Step 1.

Step 2. *Theorem 2.1.4 holds when H has compact support.*

Theorem 2.1.3 implies the statement of 2.1.4 in the case $H = 0$. Under the assumption of Step 2 the assertion of Theorem 2.1.4 with $H \neq 0$ reduces to the assertion with $H = 0$ by naturality. More precisely, define

$$\widetilde{J}_t := \phi_t^* J_t, \quad \widetilde{L}_0 := L_0, \quad \widetilde{L}_1 := \phi_1^{-1}(L_1), \quad \widetilde{K} := \bigcup_{t \in [0,1]} \phi_t^{-1}(K).$$

Then \widetilde{K} is a compact subset of M and $\widetilde{L}_0, \widetilde{L}_1 \subset M$ satisfy (H). Hence the tuple

$$(M, \omega, \widetilde{L}_0, \widetilde{L}_1, \widetilde{H} = 0, \widetilde{J}, \widetilde{\Lambda} = \Lambda, \widetilde{U} = U, \widetilde{V} = V, \widetilde{K}) \quad (2.9)$$

satisfies the hypotheses of Theorem 2.1.4 with the Hamiltonian function equal to zero. Hence, by Theorem 2.1.3, there exists a constant $\hbar > 0$ such that the result holds for this tuple.

Now let $u : [-T, T] \times [0, 1] \rightarrow M$ be a smooth function that satisfies (2.7) and (2.8) (with the above constant \hbar) and define $\widetilde{u} : [-T, T] \times [0, 1] \rightarrow M$ by

$$\widetilde{u}(s, t) := \phi_t^{-1}(u(s, t)).$$

Then \widetilde{u} satisfies (2.7) and (2.8) for the tuple (2.9). Hence

$$\widetilde{u}([-T, T] \times [0, 1]) \subset \widetilde{U} = U$$

and hence $u(s, t) \in \phi_t(U)$ for all s and t . This proves Theorem 2.1.4. \square

Theorem 2.1.5. *Assume (H) and suppose that the intersection $\Lambda = L_0 \cap L_1$ is clean. Let g be some Riemannian metric on M and let d be the distance induced by g . For every $\epsilon > 0$ there exists $\hbar > 0$ such that the following holds. For any interval $I = [a, b] \subset \mathbb{R}$ if a solution $u : I \times [0, 1] \rightarrow N$ of (2.5) satisfies the following:*

- *There exist $x, y \in \Lambda = L_0 \cap L_1$ such that*

$$\sup_t d(u(a, t), x) < \frac{\epsilon}{12} \text{ and } \sup_t d(u(b, t), y) < \frac{\epsilon}{12}.$$

- *$E(u) < \hbar$,*

then

$$u(s, t) \in B_\epsilon(x) \cap B_\epsilon(y), \quad \text{for all } (s, t) \in I \times [0, 1]. \quad (2.10)$$

Proof. See section 2.4. □

Outline of the chapter: The proofs of Theorems 2.1.1, 2.1.3 and 2.1.5 are based on some properties of holomorphic maps such as isoperimetric inequality and exponential decay. In the next sections we discuss in more details these properties. In section 2.2 we prove isoperimetric inequality in \mathbb{R}^{2n} . Then we prove that the isoperimetric inequality holds also for short curves in a symplectic manifold M with Lagrangian boundary conditions. We prove this using local Darboux charts that are adjusted to the clean intersection. Isoperimetric inequality is a crucial part of the proof of Theorem 2.1.1 as well as of the exponential decay. Theorems 2.1.3 and 2.1.5 are corollaries of the Theorem 2.1.1 and the exponential decay.

2.2 The isoperimetric inequality

The isoperimetric inequality is an inequality involving the length of a closed curve and the area enclosed by this curves. It is often expressed by saying that among all curves of given length the circle encloses the greatest area. In other words, if $\gamma \subset \mathbb{R}^2$ is simple closed curve, $A(\gamma)$ is the enclosed area and $L(\gamma)$ is its length then

$$A(\gamma) \leq \frac{1}{4\pi} L(\gamma)^2$$

and the equality holds if and only if γ is a circle. The same holds in the case that γ is a curve with endpoints on some lines through the origin. Again, the maximal area of the curvilinear triangle bounded by γ and the lines through the origin is in the case that γ is a piece of a circle.

We first prove the isoperimetric inequality in \mathbb{R}^{2n} for smooth curves with endpoints on Lagrangian planes. Let ω be the standard symplectic form in \mathbb{R}^{2n} and let $|\cdot|$ denotes the standard Euclidean norm. We define the action, length and energy of a curve γ as follows.

$$\begin{aligned}
 A(\gamma) &= \frac{1}{2} \int_0^1 \omega(\dot{\gamma}(t), \gamma(t)) dt, \\
 L(\gamma) &:= \int_0^1 |\dot{\gamma}(t)| dt, \\
 E(\gamma) &= \frac{1}{2} \int_0^1 |\dot{\gamma}(t)|^2 dt.
 \end{aligned} \tag{2.11}$$

We prove the isoperimetric inequality for curves with Lagrangian boundary conditions using Fourier series as in [16], where the analogous property was proved for closed curves.

Lemma 2.2.1. *For all smooth curves $\gamma : [0, 1] \rightarrow \mathbb{R}^{2n}$ with $\gamma(0) \in \mathbb{R}^n \times \{0\} = L_0$ and $\gamma(1) \in \mathbb{R}^d \times \{0\} \times \mathbb{R}^{n-d} = L_1$ we have that*

$$|A(\gamma)| \leq \frac{1}{\pi} L(\gamma)^2. \tag{2.12}$$

Proof. Let $\gamma : [0, 1] \rightarrow \mathbb{C}^n$, $\gamma = (z_1(t), z_2(t), \dots, z_n(t)) = (x_1(t), \dots, x_n(t), y_1(t), \dots, y_n(t))$, $0 \leq t \leq 1$, where

$$z_j(t) = x_j(t) + iy_j(t), \quad 1 \leq j \leq n.$$

Because of the boundary conditions $\gamma(0) \in \mathbb{R}^n \times 0$, $\gamma(1) \in \mathbb{R}^d \times 0 \oplus i(0 \times \mathbb{R}^{n-d})$, we have that for z_ν , $1 \leq \nu \leq d$, $z_\nu(0) \in \mathbb{R}$, $z_\nu(1) \in \mathbb{R}$. It follows that writing z_ν using the Fourier series we have

$$z_\nu(t) = \sum_{m=-\infty}^{+\infty} c_{\nu,m} e^{\pi i m t}, \quad 1 \leq \nu \leq d, \quad c_{\nu,m} \in \mathbb{R}.$$

Let γ_1 be the first d -coordinates of the curve γ i.e. $\gamma_1(t) = (z_1(t), z_2(t), \dots, z_d(t), 0, \dots, 0)$. Then

$$\gamma_1(t) = \sum_{m=-\infty}^{+\infty} v_m e^{\pi i m t}, \quad 0 \leq t \leq 1, \quad v_m = (c_{1,m}, c_{2,m}, \dots, c_{d,m}, 0, \dots, 0). \tag{2.13}$$

Similarly for $d+1 \leq \nu \leq n$, $z_\nu(0) \in \mathbb{R}$, $z_\nu(1) \in i\mathbb{R}$ it follows that

$$z_\nu(t) = \sum_{m=-\infty}^{+\infty} c_{\nu,m} e^{i(\frac{\pi}{2} + m\pi)t}, \quad d+1 \leq \nu \leq n, \quad c_{\nu,m} \in \mathbb{R}$$

Let $\gamma_2(t) = (0, \dots, 0, z_{d+1}(t), z_{d+2}(t), \dots, z_n(t))$. Then

$$\gamma_2(t) = \sum_{m=-\infty}^{+\infty} v_m e^{i(\frac{\pi}{2} + m\pi)t}, \quad 0 \leq t \leq 1, \quad v_m = (0, \dots, 0, c_{d+1,m}, c_{d+2,m}, \dots, c_{n,m}) \quad (2.14)$$

First, notice that it is enough to prove that

$$|A(\gamma)| \leq \frac{2}{\pi} E(\gamma) \quad (2.15)$$

Since, if γ is naturally parametrized then $E(\gamma) = \frac{1}{2} L(\gamma)^2$. If γ isn't parametrized by arc length, reparametrize it i.e. take a function $\alpha : [0, 1] \rightarrow [0, 1]$ such that $|\frac{d}{dt}\gamma(\alpha(t))|_e = L(\gamma)$. Then it follows from (2.15) that

$$|A(\gamma)| = |A(\gamma \circ \alpha)| \leq \frac{2}{\pi} E(\gamma \circ \alpha) = \frac{1}{\pi} L(\gamma)^2$$

It is enough to prove the inequality separately for γ_1 and γ_2 as $|A((\gamma_1, \gamma_2))| \leq |A(\gamma_1)| + |A(\gamma_2)|$ and $E((\gamma_1, \gamma_2)) = E(\gamma_1) + E(\gamma_2)$. Now the proof of Lemma 2.2.1 follows directly from Lemma 2.2.2. \square

Lemma 2.2.2. *Let γ_1 and γ_2 be paths as in (2.13) and (2.14). Then*

$$|A(\gamma_1)| \leq \frac{1}{\pi} E(\gamma_1), \quad |A(\gamma_2)| \leq \frac{2}{\pi} E(\gamma_2). \quad (2.16)$$

Proof. Let $\gamma_1(t)$ be as in (2.13), and

$$A(\gamma_1) = -\frac{1}{2} \int_0^1 \omega(\gamma_1(t), \dot{\gamma}_1(t)) dt = -\frac{1}{2} \int_0^1 \sum_{m,n=-\infty}^{+\infty} \omega(v_m e^{im\pi t}, in\pi v_n e^{in\pi t}) dt.$$

Let

$$\begin{aligned} J_{mn} &= \int_0^1 \omega(v_m e^{im\pi t}, in\pi v_n e^{in\pi t}) dt = \frac{1}{\pi} \int_0^\pi \omega(v_m e^{ims}, in\pi v_n e^{ins}) ds \\ &= \frac{1}{\pi} \left(\underbrace{\int_0^\pi \cos(ms) \cos(ns) ds}_{A_{mn}} + \underbrace{\int_0^\pi \sin(ms) \sin(ns) ds}_{B_{mn}} \right) \langle v_m, v_n \rangle. \end{aligned}$$

The last equality holds as $\omega(v_m, v_n) = 0$, because $v_m, v_n \in \mathbb{R}^d$. We compute the terms A_{mn} , B_{mn} :

$$A_{mn} = \begin{cases} \frac{\pi}{2}, & m=n \text{ or } m=-n \\ 0, & \text{otherwise} \end{cases} \quad B_{mn} = \begin{cases} \frac{\pi}{2}, & m=n \\ \frac{-\pi}{2}, & m=-n \\ 0, & \text{otherwise.} \end{cases}$$

Thus we get:

$$J_{mn} = \begin{cases} |v_n|^2, & m = n \\ 0, & \text{otherwise.} \end{cases}$$

It follows that

$$A(\gamma_1) = -\frac{1}{2} \int_0^1 \sum_{m,n=-\infty}^{+\infty} \omega(v_m e^{im\pi t}, in\pi v_n e^{in\pi t}) dt = -\frac{\pi}{2} \sum_{n=-\infty}^{+\infty} n |v_n|^2.$$

Similarly, we compute the energy $E(\gamma_1)$:

$$\begin{aligned} E(\gamma_1) &= \frac{1}{2} \int_0^1 \omega(\dot{\gamma}_1(t), i\dot{\gamma}_1(t)) dt = \frac{1}{2} \int_0^1 \sum_{m,n=-\infty}^{+\infty} \omega(v_m m\pi i e^{im\pi t}, -\pi v_n n e^{in\pi t}) dt \\ &= \frac{\pi^2}{2} \sum_{m,n=-\infty}^{+\infty} mn \int_0^1 \omega(v_m e^{im\pi t}, v_n i e^{in\pi t}) dt \\ &= \frac{\pi^2}{2} \sum_{n=-\infty}^{+\infty} n^2 |v_n|^2 \end{aligned}$$

Now it is obvious that

$$|A(\gamma_1)| \leq \frac{\pi}{2} \sum_n |n| |v_n|^2 \leq \frac{1}{\pi} E(\gamma_1)$$

Analogously we obtain for γ_2 as in (2.14)

$$|A(\gamma_2)| = \frac{\pi}{2} \sum_{n=-\infty}^{+\infty} |n + \frac{1}{2}| |v_n|^2 \leq \frac{2}{\pi} \frac{\pi^2}{2} \sum_{n=-\infty}^{+\infty} |n + \frac{1}{2}|^2 |v_n|^2 = \frac{2}{\pi} E(\gamma_2)$$

□

Remark 2.2.3. Notice that the isoperimetric inequality for curves with end points in Lagrangian plane L follows from Lemma 2.2.2. Namely, we can take $d = n$, i.e. $L_0 = L_1 = L$, then it follows from lemma 2.2.2

$$A(\gamma) \leq \frac{1}{\pi} E(\gamma) \leq \frac{1}{2\pi} L(\gamma)^2.$$

In the case that γ is a closed curve we can take a plane L_0 that divides γ into two pieces γ_i , $i = 0, 1$ of equal length. For each curve γ_i the isoperimetric inequality holds

$$A(\gamma_i) \leq \frac{1}{2\pi} L(\gamma_i)^2 = \frac{1}{8\pi} L(\gamma)^2.$$

Summing the previous inequalities for $i = 0, 1$ we obtain

$$A(\gamma) \leq \frac{1}{4\pi} L(\gamma)^2. \quad (2.17)$$

The next corollary easily follows from lemma 2.2.1.

Corollary 2.2.4. *Let $Q \subset \mathbb{R} \times [0, 1]$ be a submanifold with corners such that all the corners are contained in $\mathbb{R} \times \{0, 1\}$. Then any smooth map*

$$u : (Q, Q \cap (\mathbb{R} \times \{0\}), Q \cap (\mathbb{R} \times \{1\})) \rightarrow (\mathbb{R}^{2n}, L_0, L_1) = (\mathbb{R}^{2n}, \mathbb{R}^n \times \{0\}, \mathbb{R}^d \times \{0\} \times \mathbb{R}^{n-d}),$$

satisfies the following:

$$\int_Q u^* \omega_{std} \leq \frac{1}{\pi} \cdot L(u|_{u^{-1}(u(\partial Q) \setminus (L_0 \cup L_1))})^2 \quad (2.18)$$

where L denotes the Euclidean length.

Proof. First notice that all boundary curves $u(\partial Q)$ are of the following kind

1. closed
2. Have both endpoints in L_0 or L_1 .
3. Have one boundary point in L_0 and the other in L_1
4. Are contained in L_0 or L_1 .

Using Stoke's theorem we have:

$$\begin{aligned} \int_Q u^* \omega_{std} &= \int_Q u^* d\lambda = \int_{\partial Q} u^* \lambda = \sum_i \int_{\gamma_i} \lambda \\ &= \sum_i \int_0^1 \lambda(\gamma_i(t))(\dot{\gamma}_i(t)) dt = \sum_i \frac{1}{2} \int_0^1 \omega_{std}(\dot{\gamma}_i(t), \gamma_i(t)) dt. \end{aligned}$$

Notice that for $\gamma \subset L_i$, $i = 0, 1$ the integral $\int_0^1 \omega_{std}(\dot{\gamma}(t), \gamma(t)) = 0$, as $\omega_{std}|_{L_i} \equiv 0$, $i = 0, 1$. Thus,

$$\int_Q u^* \omega_{std} = \sum_{i \in I} \frac{1}{2} \int_0^1 \omega_{std}(\dot{\gamma}_i(t), \gamma_i(t)) dt = \sum_{i \in I} A(\gamma_i),$$

where I are the curves of type 1 – 3. In lemma 2.2.1 we have proved that

$$|A(\gamma)| \leq \frac{1}{\pi} L(\gamma)^2,$$

for any curve of the type 1-3. Thus we have :

$$\int_Q u^* \omega_{std} \leq \frac{1}{\pi} \sum_{i \in I} L(\gamma_i)^2 \leq \frac{1}{\pi} \left(\sum_{i \in I} L(\gamma_i) \right)^2.$$

□

2.2.1 Isoperimetric inequality in symplectic manifolds

The isoperimetric inequality holds not only for curves in Euclidean space, but also for short closed curves γ on symplectic manifolds or for short curves with Lagrangian boundary conditions. In both cases the area should be understood as a symplectic area. In the case of a path with Lagrangian boundaries it should be understood as the area of a curvilinear triangle with one side γ and the other two sides on the Lagrangian submanifolds. There are many curvilinear triangles with one side γ and the other two on Lagrangian submanifolds, but if γ is sufficiently short all of them will have the same symplectic area. Throughout this section we shall assume the following:

(H1) $L_0, L_1 \subset M$ are Lagrangian submanifolds of a symplectic manifold (M, ω) and the intersection $\Lambda = L_0 \cap L_1$ is clean, i.e.

$$T_p \Lambda = T_p L_0 \cap T_p L_1, \quad \forall p \in \Lambda$$

We also assume that $d = \dim(\Lambda)$.

Locally the clean intersection looks particularly nice. Namely for any point $p \in \Lambda$ there exists a local symplectic chart which maps L_0 into $\mathbb{R}^n \times \{0\}$ and L_1 into $\mathbb{R}^d \times \{0\} \times \mathbb{R}^{n-d}$. We first construct such Darboux charts adapted to clean intersections.

Lemma 2.2.5 (Darboux charts for clean intersection). *Assume (H1). Then for any $p \in L_0 \cap L_1 = \Lambda$ there exists a neighborhood $U(p)$ and a symplectomorphism $\phi : U(p) \rightarrow V(0) \subset \mathbb{R}^{2n}$ such that*

$$\begin{aligned} \phi(U(p) \cap \Lambda) &\subset \mathbb{R}^d \times \{0\}, \\ \phi(U(p) \cap L_0) &\subset \mathbb{R}^n \times \{0\}, \\ \phi(U(p) \cap L_1) &\subset \mathbb{R}^d \times \{0\} \times \mathbb{R}^{n-d}. \end{aligned} \tag{2.19}$$

Proof. Let $p \in \Lambda$, from the Lagrangian neighborhood theorem we know that there exists a neighborhood of $L_0, U(L_0)$ and a symplectomorphism $\psi : U(L_0) \rightarrow V(L_0) \subset T^*L_0$ such that $\psi(L_0) = L_0$. Choose local symplectic coordinates in a neighborhood $V(p)$ of $\psi(p) = p \in T^*L_0$, i.e. a symplectomorphism $\psi_1 : V(p) \rightarrow U(0) \subset \mathbb{R}^{2n}$, such that in these coordinates Λ, L_0 and L_1 are given as follows

$$\begin{cases} \Lambda : (x, 0, 0, \dots, 0), & x = (x_1, x_2, \dots, x_d) \\ L_0 : (x, y, 0, \dots, 0), & x = (x_1, x_2, \dots, x_d), y = (y_1, \dots, y_{n-d}) \\ L_1 : (x, f(x, z), g(x, z), z), & x = (x_1, \dots, x_d), z = (z_1, \dots, z_{n-d}) \end{cases}$$

and $f : \mathbb{R}^n \rightarrow \mathbb{R}^{n-d}, g : \mathbb{R}^n \rightarrow \mathbb{R}^d$ are differentiable maps with $f(x, 0) = g(x, 0) = 0$. Let $\Psi : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be given by

$$\Psi(x, y, u, z) = (x, y + f(x, z), u + g(x, z), z), \quad x, u \in \mathbb{R}^d, y, z \in \mathbb{R}^{n-d}. \quad (2.20)$$

Notice that $\Psi(x, y, 0, 0) = (x, y, 0, 0) \subset L_0$ and $\Psi(x, 0, 0, z) = (x, f(x, z), g(x, z), z) \subset L_1$. The mapping Ψ is a diffeomorphism locally as its derivative is given by

$$d\Psi(x, y, u, z) = \begin{pmatrix} 1 & f_x & g_x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & f_z & g_z & 1 \end{pmatrix}$$

and the determinant of $d\Psi$ is 1. The mapping Ψ is a symplectomorphism if and only if $f_z^T = f_z, g_x^T = g_x$ and $f_x + g_z^T = 0$. From the fact that L_1 is Lagrangian follows that Ψ is also a symplectomorphism. Now the desired symplectomorphism $\phi = \Psi^{-1} \circ \psi_1 \circ \psi : U(p) \rightarrow \mathbb{R}^{2n}$ and satisfies the properties (2.19). \square

The previous lemma follows also as a corollary of a more general result in the thesis of M.Pozniak ([18]). M.Pozniak proves the analog of Weinstein's neighborhood theorem for clean intersection. More precisely he proves that a neighborhood $U(\Lambda)$ is symplectomorphic to the neighborhood of $\Lambda \subset T^*L_0$ and such symplectomorphism maps L_0 into the zero section and L_1 into the conormal bundle of Λ . Here we actually need a weaker result that concerns only the local representation of clean intersection.

Remark 2.2.6. Let ϕ be a local Darboux chart constructed in Lemma 2.2.5, then the 1-form $\lambda = \phi^* \sum_{i=1}^n x_i dy_i$ satisfies $\omega = d\lambda$ and λ vanishes on TL_0 and TL_1 .

Definition 2.2.7 (Darboux radius). Assume (H) and (H1) and fix a Riemannian metric g on M . A constant $\delta > 0$ is called a **Darboux radius** for (M, L_0, L_1, Λ) if for every $p \in \Lambda$ there exists a coordinate chart $\phi : B_\delta(p) \rightarrow \mathbb{R}^{2n}$ such that

- $\phi^* \omega_{std} = \omega|_{B_\delta(p)}$
- $\phi(L_0 \cap B_\delta(p)) = \mathbb{R}^n \times \{0\} \cap \phi(B_\delta(p))$
- $\phi(L_1 \cap B_\delta(p)) = \mathbb{R}^d \times \{0\} \times \mathbb{R}^{n-d} \cap \phi(B_\delta(p))$
- $L_0 \cap B_r(p)$ and $L_1 \cap B_r(p)$ are connected for all $0 < r \leq \delta$.

Such a coordinate chart we call Darboux coordinate chart for clean intersection.

Remark 2.2.8. By Lemma 2.2.5 the set of Darboux radiuses is nonempty. Moreover it follows from the definition if δ is a Darboux radius so is every positive number $r < \delta$.

Lemma 2.2.9. Assume (H) and (H1). For every $\delta > 0$ there exists $\rho > 0$ such that the following holds. Let $\gamma : [0, 1] \rightarrow N$ be a smooth curve with $\gamma(i) \in L_i$, $i = 0, 1$. If $\ell(\gamma) < \rho$ then

$$\gamma \subset B_\delta(x_0)$$

for some $x_0 \in \Lambda$.

Proof. Let δ be an arbitrary positive number. Observe the following sets:

$$\begin{aligned} L_{0,\delta/2} &:= \{x \in L_0 \cap N : d(x, \Lambda) \geq \delta/2\} \\ L_{1,\delta/2} &:= \{x \in L_1 \cap N : d(x, \Lambda) \geq \delta/2\} \end{aligned}$$

Obviously $L_{0,\delta/2}$ and $L_{1,\delta/2}$ are compact subsets of $L_0 \cap N$ and $L_1 \cap N$. Let $\rho > 0$ be a number that satisfies

$$d(L_0, L_{1,\delta/2}) \geq \rho, \quad d(L_1, L_{0,\delta/2}) \geq \rho.$$

Obviously such a positive number exists and $\rho < \delta/2$. Now if $\ell(\gamma) < \rho$, it cannot happen that $\gamma(0) \in L_{0,\delta/2}$ and $\gamma(1) \in L_{1,\delta/2}$, as the distance between these sets is bigger than ρ . Thus, for example $d(\gamma(0), \Lambda) < \delta/2$, i.e there exists $x_0 \in \Lambda$ such that $d(x_0, \gamma(0)) \leq \delta/2$. Then $\gamma \subset B_\delta(x_0)$. \square

Lemma 2.2.10. *Assume (H) and (H1). Let $\delta > 0$ be a Darboux radius and let $x_0 \in \Lambda$. Let $\gamma : [0, 1] \rightarrow B_\delta(x_0)$ be a smooth curve such that $\gamma(i) \in L_i$, $i = 0, 1$. Denote*

$$Q = \{z \in \mathbb{C} : |z| \leq 1, \operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0\},$$

Then there exist $u : Q \rightarrow B_\delta(x_0)$ that satisfies

$$u(e^{\frac{\pi}{2}it}) = \gamma(t), \quad u(Q \cap \mathbb{R}) \subset L_0, \quad u(Q \cap i\mathbb{R}) \subset L_1$$

Proof. Let $\alpha_i : [0, 1] \rightarrow L_i \cap B_\delta(x_0)$ be such that $\alpha_i(0) = x_0$ and $\alpha_i(1) = \gamma(i)$. Such paths α_i exist as $\delta > 0$ is a Darboux radius. The loop that we obtain concatenating $\alpha_i, i = 0, 1$ and γ is contractible as it is contained in a convex neighborhood of a point. Thus there exists a desired map u . \square

Definition 2.2.11. *Assume (H) and (H1). Choose $\delta > 0$ so small that 3δ is still a Darboux radius. Let $\rho > 0$ be the constant of Lemma 2.2.9 for such a δ . Let $\gamma : [0, 1] \rightarrow M$ be a smooth path satisfying*

$$\gamma(i) \in L_i, \quad i = 0, 1, \ell(\gamma) < \rho, \quad \gamma([0, 1]) \subset N$$

*Choose $x_0 \in \Lambda$ such that $\gamma([0, 1]) \subset B_\delta(x_0)$ (see Lemma 2.2.9) and let $u : Q \rightarrow B_\delta(x_0)$ be as in Lemma 2.2.10. The **local symplectic action** of γ is a real number*

$$a(\gamma) := \int_Q u^* \omega. \tag{2.21}$$

Claim 2.2.12. *The local symplectic action is well defined, i.e. it doesn't depend on the choice of x_0 and u in the definition 2.2.11.*

Proof. Let $x_1 \in \Lambda$ be another point such that $\gamma([0, 1]) \subset B_\delta(x_1)$ and let $u' : Q \rightarrow B_\delta(x_1)$ satisfy $u'(e^{i\pi/2t}) = \gamma(t)$. Then $d(x_0, x_1) < d(x_0, \gamma(t)) + d(\gamma(t), x_1) < 2\delta$. Thus x_0 and x_1 are contained in a single Darboux chart. There exists a 1-form λ on $B_{2\delta}(x_0)$ such that $d\lambda = \omega$. Such 1-form λ has additional property that it vanishes on Lagrangian submanifolds L_0 and L_1 . Then

$$\int_Q u^* \omega = \int_Q u^* d\lambda = \int_{\partial Q} u^* \lambda = \int_0^1 \lambda(\dot{\gamma}(t)) dt.$$

Similarly we have $\int_Q (u')^* \omega = \int_0^1 \lambda(\dot{\gamma}(t)) dt$. \square

Remark 2.2.13. We see from the proof of claim 2.2.12 that it is possible to define

$$a(\gamma) = \int_{\gamma} \lambda,$$

where $\lambda \in \Omega^1(B_{\delta}(x_0))$ is 1-form which vanishes on Lagrangian submanifolds and $d\lambda = \omega|_{B_{\delta}(x_0)}$.

Now we can prove the isoperimetric inequality for short curves with Lagrangian boundary conditions.

Lemma 2.2.14 (Isoperimetric inequality). Assume (H) and (H1). There exist positive constants ρ_0 and c such that for every $\gamma : [0, 1] \rightarrow N$, $\gamma(i) \in L_i, i = 0, 1$ the following holds:
If $\ell(\gamma) < \rho_0$ then

$$a(\gamma) \leq c\ell(\gamma)^2. \quad (2.22)$$

Proof. Let δ be a Darboux radius. We can cover Λ with finitely many Darboux charts as in Definition 2.2.7 $\phi_i : B_{\delta/2}(x_i) \rightarrow \mathbb{R}^{2n}$, $x_i \in \Lambda$ such that $\|d\phi_i(x)\| \leq c'$ for some constant c' and $\forall x \in B_{\delta}(x_i)$ and for all i . Now take such ρ_0 so that the claim of the lemma 2.2.9 is satisfied with the constant $\delta_0 = \delta/2$. It follows that if $\ell(\gamma) < \rho_0$ then $\gamma \subset B_{\delta/2}(x) \subset B_{\delta}(x_i)$ for some i . As γ is contained in one Darboux chart we can use the fact that isoperimetric inequality holds in \mathbb{R}^{2n} and that the mapping $\phi = \phi_i$ preserves the $a(\gamma)$ and changes the length only up to some constant. More precisely we have

$$a(\gamma) = \int_Q u_{\gamma}^* \omega = \int_Q (\phi \circ u_{\gamma})^* w_{std} = A(\phi \circ \gamma) \leq \frac{1}{\pi} L(\phi \circ \gamma)^2$$

Here the last inequality follows from Lemma 2.2.1. On the other hand, $L(\phi \circ \gamma) \leq c'\ell(\gamma)$, thus we easily get

$$a(\gamma) \leq \frac{1}{\pi} (c')^2 \ell(\gamma)^2 = c\ell(\gamma)^2.$$

□

The analog of the Corollary 2.2.4 holds for maps $u : Q \rightarrow N$, provided that the image of u has small diameter.

Lemma 2.2.15. Assume (H) and (H1). Then there exist constants r_1 and c such that the following holds. Let $Q \subset \mathbb{R} \times [0, 1]$ be a compact 2-dimensional submanifold with corners such that the corners of Q are contained in $\mathbb{R} \times$

$\{0, 1\}$. Denote $\Gamma = \overline{\partial Q \setminus \mathbb{R} \times \{0, 1\}}$. Thus Γ is a compact 1-dimensional manifold with boundary and $\partial\Gamma \subset \mathbb{R} \times \{0, 1\}$. Let

$$u : (Q, Q \cap \mathbb{R} \times \{0\}, Q \cap \mathbb{R} \times \{1\}) \rightarrow (N, L_0, L_1),$$

be a smooth map such that $\text{diam}(u(Q)) \leq r_1$ then

$$\int_Q u^* \omega \leq c \ell^2(u|_\Gamma).$$

Proof. We distinguish the following cases

- 1) Suppose first that $Q \cap \mathbb{R} \times \{0\} \neq \emptyset$ and $Q \cap \mathbb{R} \times \{1\} \neq \emptyset$. Let $z_i \in Q \cap \mathbb{R} \times \{i\}$. Let $\delta > 0$ be so small that 2δ is still a Darboux radius. Let $\rho > 0$ be the corresponding constant as in Lemma 2.2.9 and take $r_1 < \min\{\delta, \rho\}$. As $d(u(z_0), u(z_1)) \leq r_1 < \rho$ it follows from Lemma 2.2.9 that there exists $x_0 \in \Lambda$ such that $u(z_i) \in B_\delta(x_0)$. Thus it follows that $u(Q) \subset B_{\delta+r_1}(x_0) \subset B_{2\delta}(x_0)$. We use the Darboux chart to map $u(Q)$ into \mathbb{R}^{2n} .
- 2) Suppose now that $Q \cap \mathbb{R} \times \{0\} \neq \emptyset$ and $Q \cap \mathbb{R} \times \{1\} = \emptyset$. The case $Q \cap \mathbb{R} \times \{0\} = \emptyset$ and $Q \cap \mathbb{R} \times \{1\} \neq \emptyset$ is analog. In this case we can use the Lagrangian- Darboux local chart to map $u(Q)$ into \mathbb{R}^{2n} .
- 3) If $\mathbb{R} \times \{0, 1\} \cap Q = \emptyset$ we can use the standard Darboux chart to map $u(Q)$ into \mathbb{R}^{2n} .

In any of the three mentioned cases the value of $\int_Q u^* \omega$ won't change using the symplectomorphism and the length will change only up to a constant. Hence the assertion follows from the Corollary 2.2.4.

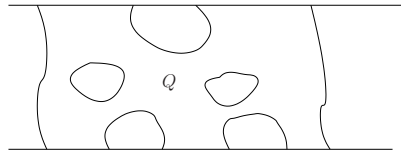


Figure 2.2: Domain Q .

□

2.3 Exponential decay

It is well known that holomorphic curves with bounded energy decay exponentially. Still, in the existing literature we weren't able to find the right reference for the case of holomorphic curves with boundary on Lagrangians that intersect cleanly and in the case that J_t is only tame almost complex structure. For the case of transverse intersection of Lagrangian submanifolds and compatible almost complex structure we refer to [19]. For the sake of completeness we include the missing case in this section.

The setup is the same as in the introduction and we shall prove that bounded energy solutions of (2.5) decay exponentially. In the case of infinite strip this means that

$$p = \lim_{s \rightarrow \infty} u(s, t)$$

exists and is an intersection point $p \in \Lambda = L_0 \cap L_1$. The convergence will be exponential with all the derivatives. In the case of finite strips $u : I \times [0, 1] \rightarrow N$ we shall see that $u(s, t)$ is close to some point in Λ for those s far away from ∂I .

To state this more precisely, for any interval $I = [a, b]$ or $I = [a, b]$ denote with $d(s, \partial I)$ the distance between s and the boundary of I . Let

$$D_r(I) := \left\{ (s, t) \in I : d(s, \partial I) \geq r, t \in [0, 1] \right\}. \quad (2.23)$$

Notice that in the case $I = [a, b]$ we have that

$$D_r([a, b]) = \begin{cases} [a + r, b - r] \times [0, 1], & r \leq (b - a)/2 \\ \emptyset, & r > (b - a)/2 \end{cases}$$

for $I = [a, +\infty)$, we have that $D_r(I) = [a + r, +\infty) \times [0, 1]$. Fix a Riemannian metric g on M and denote by d be the distance induced by g .

Proposition 2.3.1. *Assume (H) and (H1). There exists $\mu > 0$ such that for all $0 < r_0$ there exist $\delta > 0$ and c_i , $i = 0, 1, 2$ with the following properties: Assume that $u : I \times [0, 1] \rightarrow N$ is a solution of (2.5) with*

$$E(u) < \delta,$$

then u satisfies the following:

- i) $E(u|_{D_r(I)}) \leq c_0 e^{-2\mu r} E(u)$,
- ii) $|\partial_s u|_g \leq c_1 e^{-\mu r} \sqrt{E(u)}$, for all $(s, t) \in D_{r+r_0}(I)$,
- iii) $\sup_{(s_1, t_1), (s_2, t_2) \in D_{r+r_0}(I)} d(u(s_1, t_1), u(s_2, t_2)) < c_2 e^{-\mu r} \sqrt{E(u)}$

for all $r \geq r_0$.

Proof. See subsection 2.3.2. □

Direct corollaries of the previous proposition are the following:

Corollary 2.3.2. *Assume (H) and (H1). There exists a positive constant μ such that the following holds. Assume that $u : \mathbb{R}^\pm \times [0, 1] \rightarrow N$ is a solution of (2.5) with finite energy, $E(u) < +\infty$. Then*

i) *u converges toward some point $p \in \Lambda$, i.e. the limit*

$$\lim_{s \rightarrow \infty} u(s, t) = p \in \Lambda = L_0 \cap L_1.$$

ii) *The convergence is exponential, i.e. there exist positive constants r_0 , d_0 and d_1 that depend on $E(u)$ such that*

$$\begin{aligned} |\partial_s u(s, t)| &\leq d_0 e^{-\mu|s|} \\ d(u(s, t), p) &\leq d_1 e^{-\mu|s|} \end{aligned} \tag{2.24}$$

for all $|s| \geq r_0$ and all $t \in [0, 1]$.

Corollary 2.3.3. *For every $\delta > 0$ and for every $r_1 > 0$ there exist $\hbar > 0$ such that the following holds for any holomorphic curve $u : I \times [0, 1] \rightarrow N \subset M$. If $E(u) < \hbar$ then there exists $x_0 \in \Lambda$ such that*

$$d(u(s, t), x_0) < \delta, \quad (s, t) \in D_{r_1}(I) \tag{2.25}$$

Proof. From proposition 2.3.1 we have that

$$d(u(s_1, t_1), u(s_2, t_2)) < ce^{-\mu r_1} \sqrt{E(u)}$$

for all $(s_i, t_i) \in D_{r_1}(I)$, $i = 0, 1$. Taking the energy sufficiently small the right hand side of the previous inequality can be made arbitrary small. Suppose that $ce^{-\mu r_1} \sqrt{E(u)} < \rho$, where $\rho > 0$ is taken so small that the inequality $d(u(s, 0), u(s, 1)) < \rho$, $(s, 0) \in D_{r_1}(I)$ implies that there exists $x_0 \in \Lambda$ such that $d(x_0, u(s, 0)) < \delta/2$. Such ρ exists from lemma 2.2.9. We may assume w.l.o.g. that $\rho < \delta/2$. Now we have

$$d(u(s_1, t_1), x_0) < d(u(s, 0), x_0) + d(u(s, 0), u(s_1, t_1)) < \delta/2 + \rho < \delta.$$

and the previous inequality holds for all $(s_1, t_1) \in D_{r_1}(I)$. □

Proposition 2.3.4. *There exists a positive constant μ such that the following holds. Assume that $u : I \times [0, 1] \rightarrow M$ is a solution of (2.5) with finite energy $E(u) < +\infty$. Then there exist constants c_k, μ and r_k , which depend on $E(u)$ such that*

$$\|\partial_s u\|_{C^k(D_r(I))} \leq c_k e^{-\mu r} \quad (2.26)$$

for all $r \geq r_k$.

The main ingredients of the proof of Proposition 2.3.1 are the isoperimetric inequality and the mean value inequality. We have proved the isoperimetric inequality in this setup and it is left to prove the mean value inequality what we do in the next subsection.

2.3.1 The mean value inequality

The mean value inequality claims that a value of a certain function at a point can be estimated from above by its mean value.

$$f(p) \leq \frac{c}{\text{Vol}(B_r(p))} \int_{B_r(p)} f(x) dx.$$

For example subharmonic functions satisfy the mean value inequality. We shall construct a function which is not subharmonic, but whose Laplacian can be estimated from below by a polynomial of degree 2. Such a function will satisfy a generalized mean value inequality. Before defining the desired function, we construct an appropriate family of metrics adapted to the Lagrangians.

Lemma 2.3.5. *There exists a smooth family of metrics g_t such that the following holds*

- 1) L_i are totally geodesic with respect to g_i for $i = 0, 1$.
- 2) g_t is compatible with J_t for all $t \in [0, 1]$.
- 3) $J_i(p)T_p L_i$ is orthogonal complement of $T_p L_i$ for all $p \in L_i$ and for $i = 0, 1$.

Proof. In [10] is constructed a metric g_0 such that

- i) L_0 is totally geodesic with respect to g_0 .
- ii) g_0 is compatible with J_0
- iii) $J_0(p)T_p L_0$ is orthogonal complement of $T_p L_0$.

Analogously one can construct g_1 such that g_1 and L_1 satisfy the same conditions as g_0 and L_0 . Then the metric $\tilde{g}_t = (1-t)g_0 + tg_1$ satisfies 1) and 3), but not 2). Finally taking $g_t = \frac{1}{2}(\tilde{g}_t + J_t^T \tilde{g}_t J_t)$ we obtain a family of metrics g_t that satisfy also 2). \square

Let g_t be a smooth family of metrics as in Lemma 2.3.5 and let $u : I \times [0, 1] \rightarrow M$ be a J_t holomorphic curve, i.e. a solution of (2.5). We define a smooth function $e : I \times [0, 1] \rightarrow \mathbb{R}$ as follows

$$e : I \times [0, 1] \rightarrow \mathbb{R}, \quad e(s, t) = g_t(\partial_s u(s, t), \partial_s u(s, t)) = |\partial_s u|_t^2. \quad (2.27)$$

Denote with

$$H_r(s, t) = B_r(s, t) \cap (I \times [0, 1]),$$

where $B_r(s, t)$ denotes a ball of radius r centered at the point (s, t) .

Proposition 2.3.6. *Let e be as in (2.27). There exist positive constants $\tilde{\mu}$ and C such that for all $r < \frac{1}{2}$ if*

$$\int_{H_r(s, t)} e(\xi, \tau) d\xi d\tau < \tilde{\mu},$$

then

$$e(s, t) = |\partial_s u|_t \leq C(1 + \frac{1}{r^2}) \int_{H_r(s, t)} e(\xi, \tau) d\xi d\tau, \quad (2.28)$$

for all $(s, t) \in D_r(I) = \{(s, t) \in I \times [0, 1] : d(s, \partial I) \geq r, t \in [0, 1]\}$.

Proof. In Lemma 2.3.8 we prove that the function e satisfies the following inequalities :

$$\Delta e = \partial_s^2 e + \partial_t^2 e \geq -C_1(e + e^2),$$

and the normal derivative $\left| \frac{\partial e}{\partial t} \right|_{t=0,1} \leq C_2 \cdot e$, for some positive constants C_1 and C_2 . Thus the Claim follows from Theorem 2.3.7 which was proved in [28]. \square

Denote the intersection of an Euclidean ball with the half space by

$$\mathbb{D}_r(x) = B_r(x) \cap \mathbb{H}^n, \quad \mathbb{H}^n := \{(x_0, \bar{x}) : x_0 \in [0, +\infty), \bar{x} \in \mathbb{R}^{n-1}\}$$

Theorem 2.3.7. [28] *For every $n \geq 2$ there exists a constant D and for all a, b there exist $\mu(a, b) > 0$ such that the following holds: Consider (*

partial) ball $\mathbb{D}_r(y) \subset \mathbb{H}^n$ for some $r \geq 0$ and $y \in \mathbb{H}^n$. Suppose that $e \in C^2(\mathbb{D}_r(y), [0, +\infty))$ satisfies for some $A_0, A_1, B_0, B_1 \geq 0$

$$\begin{cases} \Delta e \geq -(A_0 + A_1 e + a e^{\frac{n+2}{n}}), \\ \frac{\partial}{\partial \nu}|_{\partial \mathbb{H}} e \leq B_0 + B_1 e + b e^{\frac{n+1}{n}}, \end{cases} \quad (2.29)$$

and

$$\int_{\mathbb{D}_r(y)} e \leq \mu(a, b)$$

Then

$$e(y) \leq D \left(A_0 + B_0 r + (A_1^{\frac{n}{2}} + B_1^n + r^{-n}) \int_{\mathbb{D}_r(y)} e \right).$$

We prove that the function e as in (2.27) satisfies the conditions (2.29).

Lemma 2.3.8. *Let e be as in (2.27). There exist positive constants C_1 and C_2 such that*

$$i) \quad \Delta e = \partial_s^2 e + \partial_t^2 e \geq -C_1(e + e^2).$$

$$ii) \quad \left| \frac{\partial e}{\partial t} \right|_{t=0,1} \leq C_2 \cdot e.$$

Proof. Let $\xi(s, t) = \partial_s u(s, t)$ and $\eta(s, t) = \partial_t u(s, t)$. Let ∇^t be Levi-Civita connection of the metric g_t . In order to make the notation less cumbersome, we shall write further ∇ instead of ∇^t . Because of the compatibility we have $|\xi|_t = |\eta|_t$ and as the connection is Levi-Civita we have $\nabla_t \xi = \nabla_s \eta$. Next, we have the following

$$\partial_s e(s, t) = 2g_t(\nabla_s \xi, \xi), \quad \partial_t e(s, t) = 2g_t(\nabla_t \xi, \xi) + (\partial_t g_t)(\xi, \xi)$$

and also

$$\begin{aligned} \partial_s^2 e(s, t) &= 2g_t(\nabla_s \xi, \nabla_s \xi) + 2g_t(\nabla_s \nabla_s \xi, \xi) = a_1(s, t) + a_2(s, t) \\ \partial_t^2 e(s, t) &= 2g_t(\nabla_t \xi, \nabla_t \xi) + 2g_t(\nabla_t \nabla_t \xi, \xi) + (\partial_t^2 g_t)(\xi, \xi) + 2\partial_t g_t(\nabla_t \xi, \xi) \\ &= \sum_{i=1}^4 b_i(s, t) \end{aligned}$$

The functions $a_i(s, t)$ and $b_i(s, t)$ correspond to the summands on the left side of the equalities respectively. Notice that there exist a constant \bar{c}_1 such that

$$|b_3| \leq \bar{c}_1 |\xi|_t^2, \quad |b_4| \leq \bar{c}_1 (\epsilon^2 |\nabla_t \xi|_t^2 + \frac{1}{\epsilon^2} |\xi|_t^2)$$

In order to estimate a_2 and b_2 notice that:

$$\begin{aligned}\nabla_s \nabla_s \xi + \nabla_t \nabla_t \xi &= \nabla_s (\nabla_s \xi + \nabla_t \eta) + \nabla_t \nabla_s \eta - \nabla_s \nabla_t \eta \\ &= \nabla_s (\nabla_s \xi + \nabla_t \eta) - R(\xi, \eta) \eta\end{aligned}$$

And we have

$$\begin{aligned}\nabla_s \xi + \nabla_t \eta &= \nabla_s (-J_t(u) \eta) + \nabla_t (J_t(u) \xi) \\ &= -\nabla_s (J_t(u)) \eta - J_t(u) \nabla_s \eta + \nabla_t (J_t(u)) \xi + J_t(u) \nabla_t \xi \\ &= \nabla_t (J_t(u)) \xi - \nabla_s (J_t(u)) \eta \\ &= (\partial_t J_t) \xi + (\nabla_\eta J_t) \xi - (\nabla_\xi J_t) \eta\end{aligned}$$

It follows

$$\begin{aligned}\nabla_s (\nabla_s \xi + \nabla_t \eta) &= \nabla_s \left((\partial_t J_t) \xi + (\nabla_\eta J_t) \xi - (\nabla_\xi J_t) \eta \right) \\ &= (\nabla_s \partial_t J_t(u)) \xi + \partial_t J_t(u) \nabla_s \xi + \nabla_s (\nabla_\eta J_t(u)) \xi + (\nabla_\eta J_t) \nabla_s \xi \\ &\quad - \nabla_s (\nabla_\xi J_t) \eta - (\nabla_\xi J_t) \nabla_s \eta \\ &= \sum_{i=1}^6 s_i(s, t)\end{aligned}$$

Where the terms s_i correspond to the summands in the preceding row respectively. We write $S_i = \langle s_i, \xi \rangle_{s,t}$. There exists a constant \bar{c} , which depends on g_t the almost complex structure J_t and its derivatives, such that:

$$\begin{aligned}|S_1| &\leq \bar{c} \left(|\xi|_t^2 + |\xi|_t^4 \right), \quad |S_2| \leq \bar{c} \left(\frac{1}{\epsilon^2} |\xi|_t^2 + \epsilon^2 |\nabla_s \xi|_t^2 \right), \\ |S_3| &\leq \bar{c} \left(|\xi|_t^2 + |\xi|_t^4 \right) + \bar{c} \left(\epsilon^2 |\nabla_t \xi|^2 + \frac{1}{\epsilon^2} |\xi|^4 \right), \quad |S_4| \leq \bar{c} \left(\frac{1}{\epsilon^2} |\xi|_t^4 + \epsilon^2 |\nabla_t \xi|_t^2 \right) \\ |S_5| &\leq \bar{c} \left(|\xi|_t^2 + |\xi|_t^4 \right) + \bar{c} \left(\epsilon^2 |\nabla_s \xi|^2 + \frac{1}{\epsilon^2} |\xi|^4 \right), \quad |S_6| \leq \bar{c} \left(\frac{1}{\epsilon^2} |\xi|_t^4 + \epsilon^2 |\nabla_t \xi|_t^2 \right).\end{aligned}$$

For ϵ sufficiently small we get that

$$S = |a_2| + |b_2| + |b_3| + |b_4| \leq 2|\nabla_s \xi|_t^2 + 2|\nabla_t \xi|_t^2 + C_1(|\xi|_t^2 + |\xi|_t^4),$$

for some $C_1 > 0$. Thus

$$\begin{aligned}\Delta e &= \sum_{i=1}^2 (a_i + b_i) + b_3 + b_4 \geq a_1 + b_1 - S \\ &\geq -C_1(|\xi|_t^2 + |\xi|_t^4) = -C_1(e + e^2).\end{aligned}\tag{2.30}$$

We will prove now the second inequality in lemma 2.3.8.

$$\frac{\partial e}{\partial \nu} = \frac{\partial e}{\partial t} \Big|_{t=0,1} = \left(2 \overbrace{g_t(\nabla_t \xi, \xi)}^A + \overbrace{\partial_t g_t(\xi, \xi)}^B \right) \Big|_{t=0,1} \quad (2.31)$$

$$\begin{aligned} A &= g_t(\nabla_t \xi, \xi) = g_t(\nabla_s \eta, \xi) = g_t(\nabla_s (J_t(u)\xi), \xi) \\ &= g_t(J_t(u)\nabla_s \xi, \xi) + g_t((\nabla_s J_t(u))\xi, \xi) \end{aligned}$$

Both terms in the last equality vanish at the time $t = 0, 1$. The first one vanishes as L_i , $i = 0, 1$ are totally geodesic for g_i , thus $\nabla_s \xi \in T_p L_i$, $p = u(s, i)$ and $J_i T_p L_i$ is orthogonal complement of $T_p L_i$. The second term vanishes as $\nabla_s J_t$ is skew adjoint, what follows by differentiating $g_t(J_t(u)v, w) = -g_t(v, J_t(u)w)$ in the direction of $\partial_s u$. Thus, we have that

$$\left| \frac{\partial e}{\partial \nu} \right| = \left| \frac{\partial e}{\partial t} \Big|_{t=0,1} \right| = |B| = |\partial_t g_t(\xi, \xi)| \leq C_2 |\xi|_t^2.$$

□

2.3.2 Exponential convergence

In order to prove the mean value inequality we have used special metric, but the inequality (2.28) holds no matter which metric we use to define the norm of $\partial_s u$. Until now we have mentioned three different metrics, one was just some Riemannian metric g on M , the other metric is given by pairing ω and J_t , i.e. $\langle \xi, \eta \rangle_{J_t} = \frac{1}{2} \left(\omega(\xi, J_t(\eta)) + \omega(\eta, J_t \xi) \right)$ and the third one is the metric constructed in Lemma 2.3.5. Denote with $|\cdot|_g$, $|\cdot|_{J_t}$ and $|\cdot|_t$ the norms that correspond to these metrics. As N is compact all these metrics are equivalent, i.e. there exist positive constants K_1 and K_2 such that

$$\frac{1}{K_1} |\xi|_{J_t} \leq |\xi|_t \leq K_1 |\xi|_{J_t}, \quad \frac{1}{K_2} |\xi|_g \leq |\xi|_t \leq K_2 |\xi|_g,$$

for all $t \in [0, 1]$. If the energy $E(u) = \int_{I \times [0,1]} |\partial_s u|_{J_t} ds dt$ is sufficiently small, more precisely if $E(u) \leq \mu_1 = \frac{1}{K_1^2} \tilde{\mu}$, where $\tilde{\mu}$ is the constant from Proposition 2.3.6 then:

$$\begin{aligned}
 |\partial_s u(s, t)|_g^2 &\leq K_2^2 |\partial_s u|_t^2 \leq C \cdot K_2^2 \left(1 + \frac{1}{r^2}\right) \int_{H_r(s, t)} |\partial_s u|_t^2 ds dt \\
 &= \tilde{c} \left(1 + \frac{1}{r^2}\right) \int_{H_r(s, t)} |\partial_s u|_t^2 ds dt \leq \tilde{c} K_1^2 \left(1 + \frac{1}{r^2}\right) \int_{H_r(s, t)} |\partial_s u|_{J_t}^2 ds dt \\
 &\leq c' \left(1 + \frac{1}{r^2}\right) E(u|_{H_r(s, t)}), \tag{2.32}
 \end{aligned}$$

for $(s, t) \in D_r(I)$. From here it follows that the length of the curve $\gamma_s = u(s, \cdot) : [0, 1] \rightarrow N$, $d(s, \partial I) \geq r$ is small, provided that the energy $E(u)$ is small.

Proof of the proposition 2.3.1. Let $I = [a, b]$ with $a, b \in \mathbb{R}$ (the proof is the same for infinite interval). Fix $r_0 > 0$, suppose w.l.g. that $r_0 < 1/2$. Let ρ_0 be the constant from the Lemma 2.2.14. Choose δ so small that $E(u) < \delta$ implies that the length $\ell(\gamma_s) = \ell(u(s, \cdot)) \leq \rho_0$ for all s , $d(s, \partial I) \geq r_0$. From the equation (2.32), it follows that it is enough that $E(u) < \min\{\frac{\rho_0^2}{c'(1+1/r_0^2)}, \frac{\tilde{\mu}}{K_1^2}\} = \delta$. Define

$$e(r) := E(u|_{D_r(I)}).$$

For $r \geq r_0$ we have:

$$e(r) = \iint_{D_r(I)} u^* \omega = \iint_{D_r(I)} u^* d\lambda = a(u(b-r, \cdot)) - a(u(a+r, \cdot)),$$

this holds as all the curves $\gamma_s = u(s, \cdot)$ are contained in the neighborhood where the symplectic form is exact $\omega = d\lambda$ and λ vanishes on L_0 and L_1 . Also,

$$\dot{e}(r) = - \int_0^1 |\partial_s u(b-r, t)|_{J_t}^2 dt - \int_0^1 |\partial_s u(a+r, t)|_{J_t}^2 dt.$$

Using the isoperimetric inequality, Lemma 2.2.9, and the previous two equal-

ities we get

$$\begin{aligned}
 e(r) &= a(u(b-r, \cdot)) - a(u(a+r, \cdot)) \\
 &\leq c(\ell(\gamma_{a+r})^2 + \ell(\gamma_{b-r})^2) \\
 &\leq c\left(\int_0^1 |\partial_t u(a+r, t)|_g^2 dt + \int_0^1 |\partial_t u(b-r, t)|_g^2 dt\right) \\
 &\leq c'\left(\int_0^1 |\partial_s u(b-r, t)|_{J_t}^2 dt + \int_0^1 |\partial_s u(-a+r, t)|_{J_t}^2 dt\right) \\
 &= -c'\dot{e}(r),
 \end{aligned} \tag{2.33}$$

for $r \geq r_0$. From the inequality (2.33) follows part *i*) of the Proposition 2.3.1

$$e(r) \leq e(r_0)e^{-2\mu(r-r_0)} \leq c_0 e^{-2\mu r} E(u), \tag{2.34}$$

where $c_0 = e^{2\mu r_0}$ and $\mu = \frac{1}{2c'}$. Take a point $(s, t) \in D_{r+r_0}(I)$ and a ball around that point, $B_{r_0}(s, t) \subset D_r(I)$. From the inequality (2.32) we get:

$$\begin{aligned}
 |\partial_s u(s, t)|_g^2 &\leq c'(1 + \frac{1}{r_0^2})E(u|_{H_{r_0}(s, t)}) \\
 &\leq c'(1 + \frac{1}{r_0^2})E(u|_{D_r}) = \tilde{c}_1(r_0)e(r) \\
 &\leq \tilde{c}_1(r_0)e^{-2\mu r} E(u)
 \end{aligned} \tag{2.35}$$

From here it follows *ii*) as we have

$$|\partial_s u(s, t)|_g \leq c_1 e^{-\mu r} \sqrt{E(u)}.$$

Notice that the following inequality also holds

$$\begin{aligned}
 |\partial_s u(s, t)|_g^2 &\leq c'(1 + \frac{1}{r_0^2})E(u|_{H_{r_0}(s, t)}) \leq \tilde{c} \cdot e(d(s, \partial I) - r_0) \\
 &\leq c e^{-2d(s, \partial I)} E(u),
 \end{aligned} \tag{2.36}$$

where the constant c depends on r_0 . Using the inequality (2.36) we can prove part *iii*). Let $(s_i, t_i) \in D_{r+r_0}(I)$, $i = 1, 2$. Suppose that $d(s_1, \partial I) = s_1 - a$ and $d(s_2, \partial I) = b - s_2$. We have that

$$d(u(s_1, t_1), u(s_2, t_2)) \leq \overbrace{\int_{s_1}^{s_2} |\partial_s u(s, t_1)|_g ds}^{I_1} + \overbrace{\int_{t_1}^{t_2} |\partial_t u(s_2, t)|_g dt}^{I_2}.$$

and

$$\begin{aligned}
 I_1 &= \int_{s_1}^{\frac{a+b}{2}} |\partial_s u(s, t_1)|_g ds + \int_{\frac{a+b}{2}}^{s_2} |\partial_s u(s, t_1)|_g ds \\
 &\leq c\sqrt{E(u)} \left(\int_{s_1}^{\frac{a+b}{2}} e^{-\mu(s-a)} ds + \int_{\frac{a+b}{2}}^{s_2} e^{-\mu(b-s)} ds \right) \\
 &\leq c\sqrt{E(u)} \left(\frac{e^{-\mu(s_1-a)}}{\mu} - \frac{e^{-\mu(\frac{b-a}{2})}}{\mu} + \frac{e^{-\mu(b-s_2)}}{\mu} - \frac{e^{-\mu(\frac{b-a}{2})}}{\mu} \right) \\
 &\leq c\sqrt{E(u)} \left(\frac{e^{-\mu(s_1-a)}}{\mu} + \frac{e^{-\mu(b-s_2)}}{\mu} \right) \leq \tilde{c}\sqrt{E(u)}e^{-\mu r}. \tag{2.37}
 \end{aligned}$$

The first inequality in (2.37) follows from the inequality (2.36). Similarly we have that I_2 satisfies the following.

$$\begin{aligned}
 I_2 &= \tilde{c} \int_{t_1}^{t_2} |\partial_s u(s_2, t)|_g dt \leq c\sqrt{E(u)} \int_{t_1}^{t_2} e^{-\mu(b-s_2)} dt \\
 &\leq c\sqrt{E(u)}e^{-\mu(b-s_2)} \leq c\sqrt{E(u)}e^{-\mu r} \tag{2.38}
 \end{aligned}$$

With the previous inequalities we have finished the proof of part *iii*) of Proposition 2.3.1. □

2.4 Proof of the main theorems

In this section we prove the Theorems 2.1.1, 2.1.3 and 2.1.5. The proof of the Theorem 2.1.1 is based on the fact that the isoperimetric inequality holds also for curves with Lagrangian boundary conditions what we have proved in Lemma 2.2.14 and 2.2.15.

Proof of the theorem 2.1.1. Let g be the metric on M obtained by pairing ω and J i.e. $g(\xi, \eta) = \frac{1}{2}(\omega(\xi, J\eta) + \omega(\eta, J\xi))$. Let $r_0 > 0$ be such that

$$r_0 = \min\left(\inf_{p \in N} \text{inj}(M, p), r_1\right),$$

where r_1 is the constant from Lemma 2.2.15.

There exists a constant c_1 such that for all $p \in M$ and $v, \hat{v} \in T_p M$ with $\|v\| < r_0$ we have

$$c_1 \|d \exp_p(v) \hat{v}\| \geq \|\hat{v}\|. \quad (2.39)$$

Let $\Omega \subset S = \mathbb{R} \times [0, 1]$ be a bounded open set and let $u : \Omega \rightarrow M$ be a holomorphic curve that extends continuously to $\overline{\Omega}$ and satisfies (2.2) and (2.1).

For $(s_0, t_0) \in \Omega$ denote with $p_0 = u(s_0, t_0)$. We define the function $f : \Omega \rightarrow \mathbb{R}$ as

$$f(s, t) = d(u(s, t), p_0).$$

Let

$$\Omega_r = f^{-1}([0, r)) = \{(s, t) \in \Omega \mid d(u(s, t), p_0) < r\}$$

and let

$$\Gamma_r = f^{-1}(\{r\}) = \{(s, t) \in \Omega \mid d(u(s, t), p_0) = r\}.$$

Note that $\Gamma_0 = \{(s, t) \mid u(s, t) = p_0\}$ is a finite set of points (for the proof of this fact have a look at [13] for example). Notice that $f(s, t) = \rho(u(s, t))$, where $\rho : M \rightarrow \mathbb{R}$ is given by

$$\rho(p) = d(p_0, p) = \|\exp_{p_0}^{-1} p\|.$$

and the last equality holds for points p that satisfy $d(p, p_0) < r_0$. Notice that the function f is smooth on the set $\Omega_{r_0} \setminus \Gamma_0$. Let $v = \exp_{p_0}^{-1} p$, then we have

$$\begin{aligned} |d\rho(p)(\hat{v})| &= \frac{|\langle d(\exp_{p_0}^{-1})(p)(\hat{v}), \exp_{p_0}^{-1} p \rangle|}{\sqrt{\langle \exp_{p_0}^{-1} p, \exp_{p_0}^{-1} p \rangle}} \\ &\leq \frac{\|d \exp_{p_0}^{-1} p(\hat{v})\| \|v\|}{\|v\|} \\ &= \|(d \exp_{p_0} v)^{-1}(\hat{v})\| \\ &\leq c_1 \|\hat{v}\| \end{aligned} \quad (2.40)$$

where the last inequality follows from (2.39). From the inequality (2.40) it follows that

$$\left| \frac{\partial f}{\partial s} \right| \leq c_1 \|\partial_s u\|, \quad \left| \frac{\partial f}{\partial t} \right| \leq c_1 \|\partial_t u\|. \quad (2.41)$$

and hence

$$\sqrt{\left|\frac{\partial f}{\partial s}\right|^2 + \left|\frac{\partial f}{\partial t}\right|^2} \leq \sqrt{2}c_1 \|\partial_s u\| \quad (2.42)$$

Finally define the function $a(r)$ as follows

$$a(r) := \int_{\Omega_r} u^* \omega. \quad (2.43)$$

Step 1: Let r_0 be as above and let $r < r_0$ be a regular value of the function f . Then the function a is differentiable at the point r and furthermore

$$a'(r) \geq \frac{1}{\sqrt{2}c_1} \ell(u|_{\Gamma_r}). \quad (2.44)$$

Proof. For $\delta > 0$ sufficiently small the interval $[r, r + \delta]$ consists entirely of regular values of the function f and we have

$$a(r + \delta) - a(r) = \int_{f^{-1}([r, r + \delta])} u^* \omega,$$

We reparametrize the set $f^{-1}([r, r + \delta])$ by using the gradient flow of f . Define the rescaled gradient flow ϕ by

$$\begin{aligned} \phi : \Gamma_r \times (r - \delta, r + \delta) &\rightarrow f^{-1}((r - \delta, r + \delta)), \\ \phi(\cdot, r) &= \text{Id}_{\Gamma_r}, \quad \partial_\lambda \phi(\cdot, \lambda) = \frac{\nabla f}{\|\nabla f\|^2}(\phi(\cdot, \lambda)). \end{aligned}$$

Then $f(\phi(\cdot, \lambda)) = \lambda$. Here the gradient and the norm of ∇f are understood with respect to the standard metric on \mathbb{R}^{2n} .

Suppose first that Γ_r doesn't intersect the boundary ∂S . Parametrize Γ_r by a smooth curve $\gamma_r : [0, 1] \rightarrow \Gamma_r$ and define $\psi : [0, 1] \times [r, r + \delta] \rightarrow \Omega$

$$\psi(\tau, \lambda) := \phi(\gamma_r(\tau), \lambda).$$

Orient Γ_r such that the orientations of \mathbb{R}^2 and $T_{(s,t)}\Gamma_r \oplus \mathbb{R}\nabla f(s, t)$ agree. Assume without loss of generality that γ_r is an orientation preserving diffeomorphism. Then

$$\begin{aligned}
 a(r + \delta) - a(r) &= \iint_{[0,1] \times [r, r+\delta]} \psi^* u^* \omega \\
 &= \int_r^{r+\delta} \int_0^1 u^* \omega(\partial_\tau \psi, \partial_\lambda \psi) d\tau d\lambda \\
 &= \int_r^{r+\delta} \int_0^1 \omega(du(\psi)(\partial_\tau \psi), du(\psi)\partial_\lambda \psi) \\
 &= \int_r^{r+\delta} \int_0^1 \|du(\psi)\partial_\tau \psi\| \|du(\psi)\partial_\lambda \psi\| d\tau d\lambda. \tag{2.45}
 \end{aligned}$$

Here the last equality holds because $\partial_\tau \psi$ and $\partial_\lambda \psi$ form a positive basis and u is J -holomorphic. On the other hand we have

$$\begin{aligned}
 \|du(\psi)\partial_\lambda \psi\| &= \frac{\|du(\psi)\nabla f(\psi)\|}{\|\nabla f(\psi)\|^2} \\
 &= \frac{\|\partial_s u \partial_s f + \partial_t u \partial_t f\|}{|\partial_s f|^2 + |\partial_t f|^2} \\
 &= \frac{\|\partial_s u\|}{\sqrt{|\partial_s f|^2 + |\partial_t f|^2}} \\
 &\geq \frac{1}{\sqrt{2c_1}}. \tag{2.46}
 \end{aligned}$$

The penultimate equality holds as u is J -holomorphic and hence $\partial_t u = J\partial_s u$ and the last inequality holds from the inequality (2.42). Substituting (2.46) in (2.45) we obtain

$$\begin{aligned}
 a(r + \delta) - a(r) &\geq \frac{1}{\sqrt{2c_1}} \int_r^{r+\delta} \int_0^1 \|du(\psi)\partial_\tau \psi\| d\tau d\lambda \\
 &\geq \frac{1}{\sqrt{2c_1}} \int_r^{r+\delta} \ell(u|_{\Gamma_\lambda}) d\lambda \tag{2.47}
 \end{aligned}$$

Similarly

$$a(r) - a(r - \delta) \geq \frac{1}{\sqrt{2c_1}} \int_{r-\delta}^r \ell(u|_{\Gamma_\lambda}) d\lambda$$

Dividing by $\delta > 0$ and taking limit $\delta \rightarrow 0$ we obtain

$$a'(r) \geq \frac{1}{\sqrt{2c_1}} \ell(u|_{\Gamma_r}). \tag{2.48}$$

In the case $\Gamma_r \cap \partial S \neq \emptyset$ we have that it either happens one of the situations in Figures 2.3 or 2.4 or it happens a mixture of these two cases. In the case

as in Figure 2.3 we cannot follow the flow for all time $\tau \in [0, 1]$ but only for some time period $I_\delta = [\epsilon_\delta, T_\delta] \subset [0, 1]$, more precisely

$$I_\delta = \{\tau \in [0, 1] \mid \lambda \mapsto \phi(\gamma_r(\tau), \lambda) \text{ exists on } [0, \delta]\}$$

The condition that the flow doesn't exist can be violated only for τ close to 0 and 1.

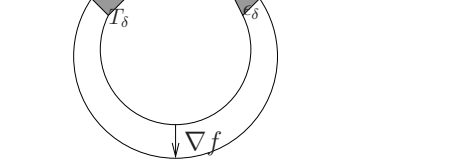


Figure 2.3: It is not possible to follow the gradient flow for all times $t \in [0, 1]$

$$a(r + \delta) - a(r) \geq \frac{1}{\sqrt{2}c_1} \int_r^{r+\delta} \int_{I_\delta} \|du(\psi)\partial_\tau\psi\| d\tau d\lambda \quad (2.49)$$

But still in the limit when $\delta \rightarrow 0$ we obtain that $a'(r) \geq \frac{1}{\sqrt{2}c_1} \ell(u|_{\Gamma_r})$ as $I_\delta \rightarrow [0, 1], \delta \rightarrow 0$. In the second case as at the Figure 2.4 we have that following the gradient flow we don't capture the whole area. In this case it still holds that

$$a(r + \delta) - a(r) \geq \frac{1}{\sqrt{2}c_1} \int_r^{r+\delta} \int_0^1 \|du(\psi)\partial_\tau\psi\| d\tau d\lambda.$$

Hence we obtain that $a'(r) \geq \frac{1}{\sqrt{2}c_1} \ell(u|_{\Gamma_r})$.

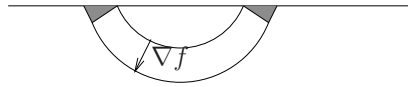


Figure 2.4: Following the gradient flow one doesn't capture the whole area

□

Step 2 : Let $a(r)$ be defined as in (2.43). There exists a constant $c_0 > 0$ such that

$$a(r) = \text{area}(u|_{\Omega_r}) \geq c_0 r^2, \quad (2.50)$$

for all $r \leq r_0$.

Proof: In Lemma 2.2.15 we have proved that the symplectic area is controlled by the length square:

$$a(t) \leq c\ell^2(u|_{\Gamma_t}).$$

Thus we have that

$$\ell(u|_{\Gamma_t}) \geq \frac{1}{\sqrt{c}} \sqrt{a(t)}. \quad (2.51)$$

Substituting the inequality (2.51) in (2.44) we get

$$(\sqrt{a(t)})' \geq \frac{1}{2\sqrt{2}\sqrt{cc_1}} = \sqrt{c_0}, \quad (2.52)$$

for all regular values $t \in [0, r_0]$. From Sard's theorem we have that the set of singular values is compact and zero measure. The function $\sqrt{a(t)}$ is monotone increasing, and hence differentiable almost everywhere. The set of regular values is open and full measure and hence can be written as a countable union of disjoint intervals $I_n = (\alpha_n, \beta_n)$. Integrating the inequality (2.52) on each interval I_n we get

$$\sqrt{a(\beta_n)} - \sqrt{a(\alpha_n)} \leq \sqrt{c_0}(\beta_n - \alpha_n) = \sqrt{c_0}\mathcal{L}(I_n).$$

As

$$\sqrt{a(r)} \geq \sum_{n=1}^{+\infty} (\sqrt{a(\beta_n)} - \sqrt{a(\alpha_n)}) \geq \sqrt{c_0} \sum_n (\beta_n - \alpha_n) \quad (2.53)$$

$$\geq \sqrt{c_0}\mathcal{L}\left(\bigcup_n I_n\right) = \sqrt{c_0}r \quad (2.54)$$

Hence, we get

$$a(r) \geq c_0 r^2 \quad (2.55)$$

for all $r \leq r_0$. \square

Proof of theorem 2.1.3. Let W be an open neighborhood of $(\Lambda \cap N) \setminus \Lambda_0$ such that $\overline{W} \cap \overline{V} = \emptyset$. Let $\widetilde{M} = \mathbb{R}^2 \times M = \mathbb{C} \times M$. We define compact sets $\widetilde{A}, \widetilde{B}, \widetilde{N} \subset \widetilde{M}$ and closed $\widetilde{L}_0, \widetilde{L}_1 \subset \widetilde{M}$ as follows

$$\begin{aligned} \widetilde{A} &= [-1, 1] \times [0, 1] \times \overline{V}, \\ \widetilde{B} &= [-1, 1] \times [0, 1] \times ((N \setminus U) \cup \overline{W}) \\ \widetilde{N} &= [-1, 1] \times [0, 1] \times N. \\ \widetilde{L}_0 &= \mathbb{R} \times \{0\} \times L_0, \quad \widetilde{L}_1 = \mathbb{R} \times \{1\} \times L_1 \end{aligned} \quad (2.56)$$

The tuple $(\widetilde{M}, \widetilde{N}, \widetilde{L}_0, \widetilde{L}_1)$ satisfies the assumptions of the Theorem 2.1.1. Let r_0 and c_0 be the corresponding constants, as in Theorem 2.1.1. We choose positive constants δ, \hbar and ϵ such that the following holds

- (i) $0 < \rho < d(\widetilde{A}, \widetilde{B})/2$ and $\rho < r_0$.
- (ii) $0 < \delta < 1$ and $\delta < \frac{c_0 \rho^2}{4}$.
- (iii) If $\gamma : [0, 1] \rightarrow N$ is a smooth curve such that $\gamma(0) \in L_0$ and $\gamma(1) \in L_1$ and

$$L(\gamma) := \int_0^1 \sqrt{\omega(\dot{\gamma}(t), J_t \dot{\gamma}(t))} dt < \epsilon$$

then $\gamma([0, 1]) \subset V \cup W$.

- (iv) If $u : [a, b] \times [0, 1] \rightarrow M$ is a J_t holomorphic curve with energy $E(u) < \hbar$ and $\frac{b-a}{2} > \delta$ then the path $u_s : [0, 1] \rightarrow M$ defined by $u_s(t) := u(s, t)$ has length $L(u_s) < \epsilon$ for $a + \delta \leq s \leq b - \delta$.
- (v) $\hbar < \frac{c_0 \rho^2}{2}$.

The existence of a constant $\epsilon > 0$ in (iii) follows from Lemma 2.2.9. The existence of a constant \hbar as in (iv) follows from mean value inequality, Proposition 2.3.6.

We claim that the assertion of Theorem 2.1.3 holds with the above constant \hbar . To see this, let $u : [a, b] \times [0, 1] \rightarrow M$ be a J_t holomorphic curve. Assume first that $\delta < \frac{b-a}{2}$. Then by (iv) we have $L(u(s, \cdot)) < \epsilon$ for $s \in (a + \delta, b - \delta)$. Hence by (iii) we have $u([a + \delta, b - \delta] \times [0, 1]) \subset V \cup W$. We claim that

$$u([a + \delta, b - \delta] \times [0, 1]) \subset V. \quad (2.57)$$

Suppose this is not the case. Since $V \cap W = \emptyset$ it would then follow that $u([a + \delta, b - \delta] \times [0, 1]) \subset W$. Define $\tilde{u} : [b - \delta, b] \times [0, 1] \rightarrow \widetilde{M}$ by

$$\tilde{u}(s, t) := (-b + s + \text{imt}, u(s, t)). \quad (2.58)$$

Then \tilde{u} takes values in \widetilde{N} and $\tilde{u}(b - \delta, t) \in \widetilde{B}$ and $\tilde{u}(b, t) \in \widetilde{A}$ for all $t \in [0, 1]$. \tilde{u} is also a \widetilde{J} holomorphic curve, where \widetilde{J} is given by

$$\widetilde{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & J_t \end{pmatrix}$$

Since $d(\widetilde{A}, \widetilde{B}) > 2\rho$, by (i), there is an element $(s_0, t_0) \in (b - \delta, b) \times [0, 1]$ such that $\tilde{p}_0 := \tilde{u}(s_0, t_0)$ satisfies

$$d(\tilde{p}_0, \widetilde{A}) > \rho, \quad d(\tilde{p}_0, \widetilde{B}) > \rho.$$

Since $\rho < r_0$, by (i), it follows from Theorem 2.1.1 that

$$E(u; [b - \delta, b] \times [0, 1]) + \delta = E(\tilde{u}; [b - \delta, b] \times [0, 1]) \geq c_0 \rho^2. \quad (2.59)$$

Hence

$$\hbar > E(u; [b - \delta, b] \times [0, 1]) \geq c_0 \rho^2 - \delta > \frac{c_0 \rho^2}{2}. \quad (2.60)$$

This contradicts (v). Thus we have proved (2.57).

Next we claim that

$$u([a, b] \times [0, 1]) \subset U. \quad (2.61)$$

If this does not hold we obtain a contradiction as above. Namely, there is a point $(s_0, t_0) \in [a, b] \times [0, 1]$ such that

$$u(s_0, t_0) \notin U.$$

By (2.57), we must have $s_0 \in (a, a + \delta) \cup (b - \delta, b)$. Suppose without loss of generality that $s_0 > b - \delta$ and define $\tilde{u} : [b - \delta, b] \times [0, 1] \rightarrow \tilde{N}$ by (2.58). Then \tilde{u} takes values in \tilde{N} and

$$\tilde{p}_0 := \tilde{u}(s_0, t_0) \in \tilde{B}, \quad \tilde{u}(b - \delta, t), \tilde{u}(b, t) \in \tilde{A},$$

for all $t \in [0, 1]$. Since $d(\tilde{A}, \tilde{B}) > \rho$ and $\rho < r_0$ by (i), it follows again from Theorem 2.1.1 that u satisfies (2.59) and (2.60), in contradiction to (v). Thus we have proved (2.61) in the case $\delta < \frac{b-a}{2}$.

If $\delta \geq \frac{b-a}{2}$ the map $\tilde{u} : [a, b] \times [0, 1] \rightarrow \tilde{M}$ defined by (2.58) takes values in \tilde{N} , has energy $E(\tilde{u}) = b - a + E(u) \leq 2\delta + \hbar < c_0 \rho^2$, and maps the set $\{a, b\} \times [0, 1]$ to \tilde{A} . Hence it follows again from Theorem 2.1.1 that its image cannot intersect \tilde{B} and so u satisfies (2.61). This proves Theorem 2.1.3. \square

Proof of the theorem 2.1.5 . Suppose that the claim isn't truth, i.e. suppose that there exists $\epsilon > 0$ and sequences $\hbar_n \rightarrow 0$ and $I_n = [a_n, b_n]$ and J_t holomorphic curves $u_n : I_n \times [0, 1]$ and points $x_n, y_n \in \Lambda$ such that

$$E(u_n) < \hbar_n, \quad \sup_t d(u_n(a_n, t), x_n) < \epsilon/12, \quad \sup_t d(u_n(b_n, t), y_n) < \epsilon/12$$

but there exist $(s_n, t_n) \in I_n \times [0, 1]$ such that $p_n = u_n(s_n, t_n) \notin B_\epsilon(x_n)$. We make the same construction as in the proof of Theorem 2.1.3, i.e. we observe $\tilde{M} = \mathbb{C} \times M$, $\tilde{N} = [-1, 1] \times [0, 1] \times N$ with Lagrangian submanifolds

$$\tilde{L}_0 = \mathbb{R} \times \{0\} \times L_0, \quad \tilde{L}_1 = (\mathbb{R} \times \{1\}) \times L_1.$$

The tuple $(\widetilde{M}, \widetilde{N}, \widetilde{L}_0, \widetilde{L}_1)$ satisfies the assumptions of the theorem 2.1.1. Let r_0 and c_0 be the constants as in Theorem 2.1.1. Let K_2 be positive constant such that

$$\frac{1}{K_2}|\xi|_g \leq |\xi|_{J_t} \leq K_2|\xi|_g. \quad (2.62)$$

Take $r = \min\{r_0, \frac{\epsilon}{24K_2}\}$ and $r_1 = c_0 \frac{r^2}{2}$.

We shall construct a new sequence $q_n = u_n(\widetilde{s}_n, \widetilde{t}_n)$ such that $B_{\frac{\epsilon}{12}}(q_n)$ contains no boundary points $u_n|_{\partial I_n \times [0,1]}$ as well as the points $u_n(s, t)$, $(s, t) \in D_{r_1}(I_n)$, where $D_{r_1}(I_n)$ is defined as in (2.23). From Corollary 2.3.3 follows that starting from some n_0 , for $n \geq n_0$ there exists $c_n \in \Lambda$ such that

$$\sup_{s,t \in D_{r_1}(I_n)} d(u_n(s, t), c_n) < \frac{\epsilon}{12}.$$

One of the following cases occurs

- 1) $y_n, c_n \notin B_{\epsilon/6}(p_n)$. Then $q_n = p_n, \widetilde{s}_n = s_n, \widetilde{t}_n = t_n$ and $B_{\epsilon/12}(q_n)$ doesn't contain points $u_n(s, t)$ for $(s, t) \in D_{r_1}(I_n)$ and $u_n(s, t)$, $(s, t) \in \partial I_n \times [0, 1]$
- 2) $y_n, c_n \in B_{\epsilon/6}(p_n) \cup B_{\epsilon/6}(x_n)$. Then $B_{\epsilon/4}(p_n) \cup B_{\epsilon/4}(x_n)$ contains all points $u_n(s, t)$, $(s, t) \in D_{r_1}(I_n)$ and all boundary points $u|_{\partial I_n \times [0,1]}$. In this case there exists a point $q_n = u_n(\widetilde{s}_n, \widetilde{t}_n)$ such that $d(q_n, x_n) \geq \frac{\epsilon}{3}$ and $d(q_n, p_n) = \epsilon/2$ as at the Figure 2.5.

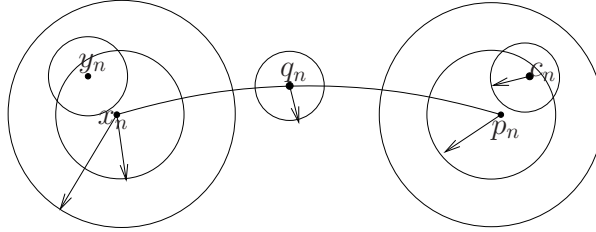


Figure 2.5:

Such a point exists as the image of u contains points from $B_{\frac{\epsilon}{12}}(x_n)$ and also p_n , thus the ball centered at p_n with radius $\frac{\epsilon}{2}$ will intersect its image. Obviously $B_{\epsilon/12}(q_n)$ contains no boundary points and no points $u(s, t)$, $(s, t) \in D_{r_1}(I_n)$.

- 3) If for example $y_n \in B_{\epsilon/6}(p_n)$ and $c_n \notin B_{\epsilon/6}(p_n) \cup B_{\epsilon/6}(x_n)$ (the reverse is equivalent), then choose again points (s'_n, t'_n) , $q'_n = u_n(s'_n, t'_n)$ such

that $d(q'_n, p_n) = \frac{5\epsilon}{12}$ and $d(q'_n, x_n) \geq \frac{\epsilon}{2}$. Then we have two cases

- a) If $c_n \notin B_{\epsilon/6}(q'_n)$ then $q_n = q'_n$, $\tilde{s}_n = s'_n$ and $\tilde{t}_n = t'_n$.
- b) If $c_n \in B_{\epsilon/6}(q'_n)$, then choose points (s''_n, t''_n) , $q''_n = u_n(s''_n, t''_n)$ such that $d(q''_n, q'_n) = \frac{\epsilon}{3}$ and $d(q''_n, x_n) \geq \epsilon/6$. Then $B_{\epsilon/12}(q''_n)$ contains no boundary points as well as points $u(s, t)$, $(s, t) \in D_{r_1}(I_n)$ and $q_n = q''_n$ and $\tilde{s}_n = s''_n$, $\tilde{t}_n = t''_n$.

In the case $\tilde{s}_n \in (a_n, a_n + r_1)$ we define $\tilde{u}_n : [a_n, a_n + r_1] \times [0, 1] \rightarrow \tilde{N}$ by $\tilde{u}_n(s, t) = (-a_n + s + \text{imt}, u_n(s, t))$, otherwise if $\tilde{s}_n \in (b_n - r_1, b_n)$ we define $\tilde{u}_n : [b_n - r_1, b_n] \times [0, 1] \rightarrow \tilde{N}$ by $\tilde{u}_n(s, t) = (-b_n + s + \text{imt}, u_n(s, t))$. Suppose that $\tilde{s}_n \in (a_n, a_n + r_1)$, the other case is analog. We define $\tilde{q}_n = (-a_n + \tilde{s}_n + \text{imt}_n, q_n)$. Let $s = a_n$ or $s = a_n + r_1$ then the distance

$$d(\tilde{q}_n, \tilde{u}_n(s, t)) = \inf\{l(\gamma) : \gamma : [0, 1] \rightarrow \tilde{M}, \gamma(0) = \tilde{q}_n, \gamma(1) = \tilde{u}_n(s, t)\}.$$

Any such curve γ can be written as $\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{C} \times M$. Then

$$\begin{aligned} \ell(\gamma) &\geq \frac{1}{2} \left(\int_0^1 |\dot{\gamma}_1(t)|_e dt + \int_0^1 |\dot{\gamma}_2(t)|_{J_t} dt \right) \\ &\geq \frac{1}{2} \int_0^1 |\dot{\gamma}_2(t)|_{J_t} dt \geq \frac{1}{2K_2} \int_0^1 |\dot{\gamma}_2(t)|_g dt \\ &\geq \frac{1}{2K_2} d(u_n(s, t), q_n) = \frac{\epsilon}{24K_2} \geq r \end{aligned}$$

Thus, we see that $\tilde{u}_n^{-1}(B_r(\tilde{q}_n))$ does not intersect the set $\{a_n\} \times [0, 1] \cup \{a_n + r_1\} \times [0, 1]$. From the monotonicity theorem for \tilde{J} holomorphic curves, Theorem 2.1.1, we have

$$A = \text{Area}(\tilde{u}_n|_{\tilde{u}_n^{-1}(B_r(\tilde{q}_n))}) \geq c_0 r^2.$$

On the other hand

$$\begin{aligned} A &= \int_{\tilde{u}_n^{-1}(B_r(\tilde{q}_n))} \tilde{u}_n^* \tilde{\omega} = \int_{\tilde{u}_n^{-1}(B_r(\tilde{q}_n))} \omega_{std} + \int_{\tilde{u}_n^{-1}(B_r(\tilde{p}_n))} u_n^* \omega \\ &\leq \int_0^1 \int_{a_n}^{a_n+r_1} \omega_{std} + E(u_n) \\ &\leq r_1 + E(u_n). \end{aligned}$$

Therefore we have that

$$c_0 r^2 \leq A \leq \frac{c_0 r^2}{2} + E(u_n),$$

what is a contradiction as $\lim_{n \rightarrow +\infty} E(u_n) = 0$.

□

Chapter 3

Fredholm theory on truncated surfaces

This chapter is entirely devoted to the study of various properties of some linear operators on truncated surfaces. These surfaces are essentially similar to half-infinite or finite strips. We shall consider first the linearized operator D_A of the form

$$D_A \xi = \partial_s \xi + A \xi,$$

where the operator A is bijective and self-adjoint and s independent. In order to establish Fredholm properties on truncated surfaces we decided to work with Hilbert spaces unlike most authors do. Thus, the domain of the operator D_A will be some $W^{2,2}$ space on strip with some boundary conditions. This approach has various advantages. One of the most important facts is that the trace spaces of such Hilbert spaces have particularly nice form and they can be described in terms of the domain of some power of the operator A . The operator D_A will not be Fredholm if we take its domain to be the mentioned Hilbert space, as it will have infinite dimensional kernel, but reducing its domain by fixing some boundary conditions which can be expressed in terms of positive and negative eigenvector spaces of the operator A we can prove that it is actually bijective. We allow later that the operator A depends on time s , thus we consider the linearized operator D of the form

$$D \xi = \partial_s \xi + A(s) \xi.$$

The operator $A(s)$ is such that the difference $D - D_A = K$ is a compact operator. Thus the operator D as a compact perturbation of the operator D_A inherits most of its properties. Our final goal will be to prove surjectivity of the operator D .

3.1 Linear estimates in abstract setting

3.1.1 (Hilbert triple and the operator in the time independent case). In this section we prove some linear elliptic estimates that will be crucial for the proof of Theorem 4.1.8. and 4.1.7. We will not specify Hilbert spaces in which we work, as we shall allow later various applications of the results of this chapter. The approach is similar to [14] and [22].

Consider the following three Hilbert spaces $H^2 \subset H^1 \subset H^0$ such that each inclusion is compact, dense and continuous. Throughout this section we shall assume the following hypothesis

(HA) Let $A : H^1 \rightarrow H^0$ be a linear, bijective, and self-adjoint operator with the domain of A^2 equal to H^2 , where

$$\text{Dom}(A^2) = \{\xi \in H^1 | A(\xi) \in H^1\}.$$

If A satisfies (HA), then there exist positive constants C_j , $j = 1, 2$ such that the following inequality holds for all $\xi \in H^j$

$$\frac{1}{C_j} \|\xi\|_{H^j} \leq \|A\xi\|_{H^{j-1}} \leq C_j \|\xi\|_{H^j}, \quad j = 1, 2. \quad (3.1)$$

The right side of the previous inequality follows from Hellinger-Toplitz theorem, and the left side from the open mapping theorem.

3.1.2 (Intended Application). We shall apply the results from this section to the following three Hilbert spaces

$$\begin{aligned} H^0 &= L^2([0, 1]) \\ H^1 &= H_{bc}^1([0, 1]) = \left\{ \xi \in H^1([0, 1], \mathbb{R}^{2n}) \mid \xi(i) \in \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \right\} \end{aligned} \quad (3.2)$$

The Hilbert space H^1 will be actually the domain of some operator A of the form $A = J_0 \partial_t + S(t)$. In that case the space H^2 will be just the domain of A^2 . In the proof of Theorem 4.1.8, the Hilbert space H^2 will be

$$H^2 = H_{bc}^2([0, 1]) = \left\{ \xi \in H^2([0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(i) \in \mathbb{R}^n \times \{0\}, \\ \partial_t \xi(i) \in \{0\} \times \mathbb{R}^n, \quad i = 0, 1 \end{array} \right\}. \quad (3.3)$$

Another typical example of such triplet is also a Gelfand triple $V \subset H \subset V^*$.

3.1.3 (Intermediate Hilbert spaces and projections). Assume (HA). Notice that each $\xi \in H^0$ can be uniquely written in the form $\xi = \sum_i x_i e_{\lambda_i}$,

where e_{λ_i} are the eigenvectors of the operator A forming an orthonormal basis of H^0 . Let

$$E^{1/2} := [H^1, H^0]_{1/2}, \quad E^{3/2} := [H^2, H^1]_{1/2}$$

(for more details on interpolation spaces we refer to Appendix 4.6). As $H^2 = \text{Dom}(A^2)$ it follows that

$$E^{3/2} = \text{Dom}(|A|^{3/2}) = \left\{ \xi = \sum_i x_i e_{\lambda_i} \mid \sum_i |\lambda_i|^3 |x_i|^2 < +\infty \right\}. \quad (3.4)$$

and analogously can be described $E^{1/2}$

$$E^{1/2} = \text{Dom}(|A|^{1/2}) = \left\{ \xi = \sum_i x_i e_{\lambda_i} \mid \sum_i |\lambda_i| |x_i|^2 < +\infty \right\}. \quad (3.5)$$

For $\xi \in E^{3/2}$ we denote by $\|\xi\|_{3/2}$ the norm of $\xi = \sum_i x_i e_{\lambda_i}$, we have

$$\|\xi\|_{3/2}^2 := \sum_i |\lambda_i|^3 |x_i|^2.$$

Let $E^\pm \subset H^0$ be positive and negative eigenvector spaces

$$E^\pm := \left\{ \xi = \sum_{\pm \lambda_i > 0} x_i e_{\lambda_i} \in H^0 \right\}.$$

Then the Hilbert space $E^{3/2}$ decomposes as

$$E^{3/2} = (E^+ \cap E^{3/2}) \oplus (E^- \cap E^{3/2}) \quad (3.6)$$

Each $\xi \in E^{3/2}$ can be written uniquely in the form $\xi = \xi^+ + \xi^-$, where $\xi^\pm \in E^\pm \cap E^{3/2}$. Denote with π^\pm the projections

$$\pi^\pm : E^{3/2} \rightarrow E^\pm \cap E^{3/2}. \quad (3.7)$$

It is crucial that $\text{Dom}(A^2) = H^2$, otherwise we would not have the decomposition of $E^{3/2}$ into its positive and negative subspaces and the projection would not be well defined. We have an analogous projection

$$\pi^\pm : E^{1/2} \rightarrow E^\pm \cap E^{1/2}. \quad (3.8)$$

3.1.4 (Hilbert spaces with strip-like domains). Let I be an interval in $\overline{\mathbb{R}}$ and let $H^2 \subset H^1 \subset H^0$ be as in 3.1.1. Observe the following three Hilbert spaces

$$\begin{aligned}\mathcal{W}^0(I) &= L^2(I, H^0) \\ \mathcal{W}^1(I) &= L^2(I, H^1) \cap W^{1,2}(I, H^0) \\ \mathcal{W}^2(I) &= L^2(I, H^2) \cap W^{1,2}(I, H^1) \cap W^{2,2}(I, H^0)\end{aligned}$$

Equip $\mathcal{W}^0(I)$, $\mathcal{W}^1(I)$ and $\mathcal{W}^2(I)$ with the following norms

$$\begin{aligned}\|\xi\|_{\mathcal{W}^0(I)}^2 &:= \int_I \|\xi\|_{H^0}^2 ds \\ \|\xi\|_{\mathcal{W}^1(I)}^2 &:= \int_I \|\xi\|_{H^1}^2 ds + \int_I \|\partial_s \xi\|_{H^0}^2 ds \\ \|\xi\|_{\mathcal{W}^2(I)}^2 &:= \int_I \|\xi\|_{H^2}^2 ds + \int_I \|\partial_s \xi\|_{H^1}^2 ds + \int_I \|\partial_s^2 \xi\|_{H^0}^2 ds.\end{aligned}$$

As the inclusion $H^i \hookrightarrow H^{i-1}$, $i = 1, 2$ is continuous it follows that there exist constants c', c'' such that

$$\|\xi\|_{\mathcal{W}^0(I)} \leq c' \|\xi\|_{\mathcal{W}^1(I)} \leq c'' \|\xi\|_{\mathcal{W}^2(I)}$$

holds for all $I \subset \overline{\mathbb{R}}$.

3.1.5. Notice that in the case H^i , $i = 0, 1$ are as in (3.2), then the Hilbert space $\mathcal{W}^1(I)$ is isometric to the following space

$$H_{bc}^1(I \times [0, 1]) := \left\{ \xi \in W^{1,2}(I \times [0, 1], \mathbb{R}^{2n}) \mid \xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \right\}. \quad (3.9)$$

In the case that $H^2 = \text{Dom}(A)^2$ the space $\mathcal{W}^2(I)$ is isometric to the following Hilbert space

$$H_{bc}^2(I \times [0, 1]) := \left\{ \xi \in H^2(I \times [0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \\ A\xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \end{array} \right\}. \quad (3.10)$$

In particular, if the space H^2 is given by (3.3) then the space $\mathcal{W}^2(I)$ is

$$W_{bc}^{2,2}(I \times [0, 1]) := \left\{ \xi \in H^2(I \times [0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \\ \partial_t \xi(s, i) \in \{0\} \times \mathbb{R}^n, i = 0, 1 \end{array} \right\}. \quad (3.11)$$

The Hilbert space $\mathcal{W}^0(I)$ is just the standard $L^2(I \times [0, 1], \mathbb{R}^{2n})$.

Observe the following linear operator

$$\begin{aligned} D_A : \mathcal{W}^i(I) &\rightarrow \mathcal{W}^{i-1}(I), \quad i = 1, 2 \\ D_A(\xi) &= \partial_s \xi(s, t) + A\xi(s, t), \end{aligned} \quad (3.12)$$

where A satisfies the assumption (HA).

Theorem 3.1.6. *Let $i = 1$ or $i = 2$ and let D_A be defined as in (3.12). Let $E^{i-1/2}, E^\pm$ and π^\pm be as in 3.1.3 and let $\mathcal{W}^i(I)$ be as in 3.1.4.*

i) *There exists a constant $c_i > 0$ such that for any interval $I = [a, b]$ and for all $\xi \in \mathcal{W}^i([a, b])$ the following inequality holds*

$$\|\xi\|_{\mathcal{W}^i([a, b])} \leq c_i \left(\|D_A \xi\|_{\mathcal{W}^{i-1}([a, b])} + \|\pi^+(\xi(a))\|_{i-\frac{1}{2}} + \|\pi^-(\xi(b))\|_{i-\frac{1}{2}} \right) \quad (3.13)$$

Furthermore the mapping

$$\begin{aligned} F : \mathcal{W}^i([a, b]) &\rightarrow \mathcal{W}^{i-1}([a, b]) \times (E^+ \cap E^{i-\frac{1}{2}}) \times (E^- \cap E^{i-\frac{1}{2}}) \\ F(\xi) &= (D_A \xi, \pi^+(\xi(a)), \pi^-(\xi(b))) \end{aligned} \quad (3.14)$$

is bijective.

ii) *The maps*

$$\begin{aligned} F^\pm : \mathcal{W}^i(\mathbb{R}^\pm) &\rightarrow \mathcal{W}^{i-1}(\mathbb{R}^\pm) \times E^\pm \cap E^{i-\frac{1}{2}} \\ F^\pm(\xi) &= (D_A \xi, \pi^\pm(\xi(0, \cdot))) \end{aligned} \quad (3.15)$$

are bijective. There exists a constant $c_i > 0$ such that for all $\xi \in \mathcal{W}^i(\mathbb{R}^\pm)$ the following inequality holds

$$\|\xi\|_{\mathcal{W}^i(\mathbb{R}^\pm)} \leq c_i \left(\|D_A \xi\|_{\mathcal{W}^{i-1}(\mathbb{R}^\pm)} + \|\pi_\pm(\xi(0, \cdot))\|_{i-\frac{1}{2}} \right). \quad (3.16)$$

Proof. We prove the theorem in the next four steps.

Step 1. Proof of the inequality (3.13) in the case $i = 1$.

$$\begin{aligned} \int_a^b \|D_A \xi\|_{H^0}^2 ds &= \int_a^b \langle \partial_s \xi + A\xi, \partial_s \xi + A\xi \rangle_{H^0} ds \\ &= \int_a^b \left(\|\partial_s \xi\|_{H^0}^2 + \|A\xi\|_{H^0}^2 \right) ds + \int_a^b \partial_s \langle \xi, A\xi \rangle_{H^0} ds. \end{aligned} \quad (3.17)$$

Thus $L = \int_a^b (\|\partial_s \xi\|_{H^0}^2 + \|A\xi\|_{H^0}^2) ds$ satisfies

$$L = \int_a^b \|D_A \xi\|_{H^0}^2 ds - \langle \xi(b), A\xi(b) \rangle_{H^0} + \langle \xi(a), A\xi(a) \rangle_{H^0}$$

From the inequality (3.1) and the previous equality we obtain

$$\begin{aligned} \|\xi\|_{\mathcal{W}^1([a,b])}^2 &\leq C_1^2 L = C_1^2 \left(\|D_A \xi\|_{\mathcal{W}^0([a,b])}^2 - \langle \xi(b), A\xi(b) \rangle_{H^0} + \langle \xi(a), A\xi(a) \rangle_{H^0} \right) \\ &\leq c_1 \left(\|D_A \xi\|_{\mathcal{W}^0([a,b])}^2 + \|\pi^+(\xi(a, \cdot))\|_{1/2}^2 + \|\pi^-(\xi(b, \cdot))\|_{1/2}^2 \right). \end{aligned} \quad (3.18)$$

In the previous inequality $\|\cdot\|_{1/2}$ norm of a $\xi(a, \cdot) = \sum_\lambda x_\lambda e_\lambda$ is given by

$$\|\xi(a, \cdot)\|_{1/2}^2 = \sum_\lambda |\lambda| |x_\lambda|^2$$

thus $\|\pi^+(\xi(a, \cdot))\|_{1/2}^2 = \sum_{\lambda>0} \lambda |x_\lambda|^2$ and analogously is given the norm of $\|\pi^-(\xi(b, \cdot))\|_{1/2}$.

Step 2. Proof of the inequality (3.13) in the case $i = 2$.

To shorten the notation we shall write $\|\cdot\|_{\mathcal{W}^i}$, $i = 0, 1, 2$ for $\|\cdot\|_{\mathcal{W}^i([a,b])}$. Substituting $A\xi$ in the inequality (3.18) we obtain

$$\begin{aligned} \|A\xi\|_{\mathcal{W}^1}^2 &\leq c_1 \left(\|D_A(A\xi)\|_{\mathcal{W}^0}^2 + \|\pi^+(A\xi(a))\|_{1/2}^2 + \|\pi^-(A\xi(b))\|_{1/2}^2 \right) \\ &\leq c_1 \left(\|A(D_A \xi)\|_{\mathcal{W}^0}^2 + \|\pi^+(A\xi(a))\|_{1/2}^2 + \|\pi^-(A\xi(b))\|_{1/2}^2 \right) \\ &\leq c'_1 \left(\|D_A \xi\|_{\mathcal{W}^1}^2 + \|\pi^+(\xi(a, \cdot))\|_{3/2}^2 + \|\pi^-(\xi(b, \cdot))\|_{3/2}^2 \right). \end{aligned} \quad (3.19)$$

The last inequality follows from (3.1) and the following observation

$$\|A\eta\|_{\mathcal{W}^0}^2 = \int_a^b \|A\eta\|_{H^0}^2 ds \leq C_1^2 \int_a^b \|\eta\|_{H^1}^2 ds \leq C_1^2 \|\eta\|_{\mathcal{W}^1}^2.$$

As the embedding $H^1 \hookrightarrow H^0$ is continuous and (3.1) holds we have

$$\|\xi\|_{H^0} \leq c' \|\xi\|_{H^1} \leq c' C_1 \|A\xi\|_{H^0} \leq c'' \|A\xi\|_{H^1}$$

Integrating the previous inequality on interval $[a, b]$ and using (3.19) we obtain

$$\|\xi\|_{\mathcal{W}^0}^2 \leq c'_2 \left(\|D_A \xi\|_{\mathcal{W}^1}^2 + \|\pi^+(\xi(a, \cdot))\|_{3/2}^2 + \|\pi^-(\xi(b, \cdot))\|_{3/2}^2 \right). \quad (3.20)$$

We have that $\|\partial_s \xi\|_{H^1}^2 \leq 2(\|D_A \xi\|_{H^1}^2 + \|A\xi\|_{H^1}^2)$. Integrating this inequality we obtain that

$$\begin{aligned} \int_a^b \|\partial_s \xi\|_{H^1}^2 ds &\leq 2 \left(\int_a^b \|D_A \xi\|_{H^1}^2 ds + \int_a^b \|A\xi\|_{H^1}^2 ds \right) \\ &\leq c_3 \left(\|D_A \xi\|_{W^1}^2 + \|A\xi\|_{W^1}^2 \right) \\ &\leq c_4 \left(\|D_A \xi\|_{W^1}^2 + \|\pi^+(\xi(a, \cdot))\|_{3/2}^2 + \|\pi^-(\xi(b, \cdot))\|_{3/2}^2 \right). \end{aligned} \quad (3.21)$$

The last inequality of (3.21) follows from (3.19). Finally, the following inequality also holds

$$\|\partial_s^2 \xi\|_{H^0}^2 \leq 2 \left(\|\partial_s(D_A \xi)\|_{H^0}^2 + \|\partial_s(A\xi)\|_{H^0}^2 \right).$$

Integrating this inequality on the interval $[a, b]$ we obtain

$$\begin{aligned} \int_a^b \|\partial_s^2 \xi\|_{H^0}^2 ds &\leq 2 \left(\|D_A \xi\|_{W^1}^2 + \|A\xi\|_{W^1}^2 \right) \\ &\leq c_5 \left(\|D_A \xi\|_{W^1}^2 + \|\pi^+(\xi(a, \cdot))\|_{3/2}^2 + \|\pi^-(\xi(b, \cdot))\|_{3/2}^2 \right). \end{aligned} \quad (3.22)$$

The inequality (3.22) follows from (3.19). Summing the inequalities (3.20), (3.21) and (3.22), we obtain

$$\|\xi\|_{W^2([a,b])} \leq c_2 \left(\|D_A \xi\|_{W^1([a,b])} + \|\pi^+(\xi(a))\|_{3/2} + \|\pi^-(\xi(b))\|_{3/2} \right). \quad (3.23)$$

Thus we have proved the inequality (3.13). This inequality implies that the mapping F is injective and has closed range. We still have to prove that it is surjective.

Step 3. Surjectivity of the operator F .

We prove surjectivity in the case $i = 2$. The proof of surjectivity in the case $i = 1$ is analogous.

Let $\eta \in \mathcal{W}^1([a, b])$ and $\zeta^\pm \in E^\pm \cap E^{3/2}$. We prove the existence of $\xi \in \mathcal{W}^2([a, b])$ which satisfies:

$$D_A \xi = \partial_s \xi(s, \cdot) + A\xi(s, \cdot) = \eta, \quad \pi^+(\xi(a, t)) = \zeta^+, \quad \pi^-(\xi(b, t)) = \zeta^- \quad (3.24)$$

Let $\xi = \sum_\lambda \xi_\lambda(s, \cdot)$ and $\eta = \sum_\lambda \eta_\lambda(s, t)$ and $\zeta^\pm = \sum_{\pm\lambda>0} \zeta_\lambda$, where ξ_λ , η_λ and ζ_λ are the eigenvectors of the operator A . In order to find the solution of

(3.24) we need to solve the following equations

$$\begin{aligned}\partial_s \xi_\lambda(s, \cdot) + \lambda \xi_\lambda(s, \cdot) &= \eta_\lambda(s, \cdot), \text{ for all } \lambda \\ \xi_\lambda(a, \cdot) &= \zeta_\lambda(\cdot), \quad \lambda > 0 \\ \xi_\lambda(b, \cdot) &= \zeta_\lambda(\cdot), \quad \lambda < 0\end{aligned}\tag{3.25}$$

The solutions ξ_λ of the previous equation is given by

$$\begin{aligned}\xi_\lambda(s, t) &= \int_a^s \eta_\lambda(y, t) e^{-\lambda(s-y)} dy + e^{-\lambda(s-a)} \zeta_\lambda(t), \quad \lambda > 0 \\ \xi_\lambda(s, t) &= e^{\lambda(b-s)} \zeta_\lambda(t) - \int_s^b \eta_\lambda(y, t) e^{\lambda(y-s)} dy, \quad \lambda < 0.\end{aligned}\tag{3.26}$$

It is left to prove that $\xi \in \mathcal{W}^2([a, b])$ what is, because of (3.1), equivalent to the following

$$\begin{aligned}\sum_\lambda \lambda^4 \int_a^b \|\xi_\lambda\|_{H^0}^2 ds &< +\infty \\ \sum_\lambda \lambda^2 \int_a^b \|\partial_s \xi_\lambda\|_{H^0}^2 ds &< +\infty \\ \sum_\lambda \int_a^b \|\partial_s^2 \xi_\lambda\|_{H^0}^2 ds &< +\infty\end{aligned}$$

Remember that $\eta \in \mathcal{W}^1([a, b])$ and $\zeta \in E$ what is equivalent to the following

$$\begin{aligned}\sum_\lambda \lambda^2 \int_a^b \|\eta_\lambda\|_{H^0}^2 ds &< +\infty \text{ and } \sum_\lambda \int_a^b \|\partial_s \eta_\lambda\|_{H^0}^2 ds < +\infty \\ \sum_\lambda |\lambda|^3 \|\zeta_\lambda\|_{H^0}^2 &< +\infty\end{aligned}\tag{3.27}$$

As ξ_λ satisfies the equation (3.25) and η_λ satisfy (3.27) it is enough to prove

$$\sum_\lambda \lambda^4 \|\xi_\lambda\|_{\mathcal{W}^0([a, b])}^2 = \sum_\lambda \lambda^4 \int_a^b \|\xi_\lambda\|_{H^0}^2 ds < +\infty.\tag{3.28}$$

Write $\xi_\lambda = v_\lambda + w_\lambda$, where

$$\begin{aligned}v_\lambda(s, t) &= \int_a^s \eta_\lambda(y, t) e^{-\lambda(s-y)} dy, \quad \lambda > 0 \\ w_\lambda(s, t) &= e^{-\lambda(s-a)} \zeta_\lambda, \quad \lambda > 0\end{aligned}$$

and analogously for $\lambda < 0$. Notice that $v_\lambda = K_\lambda * \chi_{[a,s)} \cdot \eta_\lambda$, where

$$K_\lambda(s, t) = \begin{cases} e^{-\lambda s}, & s \geq 0 \\ 0, & s < 0 \end{cases}$$

for $\lambda > 0$. Obviously $\|K_\lambda\|_{L^1} = \frac{1}{\lambda}$. Denote by $d\mu_s(\lambda) = e^{-\lambda(s-y)}\chi_{[a,s)}(y)dy$. Then

$$\int_a^b d\mu_s(y) \leq \frac{1}{\lambda}, \quad \forall s \in (a, b).$$

We can apply Jensen's inequality

$$\|v_\lambda(s, \cdot)\|_{H^0} \leq \int_a^b e^{-\lambda(s-y)}\chi_{[a,s)}\|\eta_\lambda\|_{H^0}dy = K_\lambda * f_\lambda,$$

where $f_\lambda = \chi_{[a,s)}\|\eta_\lambda\|_{H^0}$. From Young's inequality we obtain

$$\|v_\lambda\|_{\mathcal{W}^0([a,b])} \leq \|K_\lambda\|_{L^1}\|\eta_\lambda\|_{\mathcal{W}^0([a,b])} \leq \frac{1}{\lambda}\|\eta_\lambda\|_{\mathcal{W}^0([a,b])}.$$

Thus it follows from (3.27) that

$$\sum_\lambda \lambda^4 \int_a^b \|v_\lambda\|_{H^0}^2 ds \leq \sum_\lambda \lambda^2 \int_a^b \|\eta_\lambda\|_{H^0}^2 ds < +\infty. \quad (3.29)$$

On the other hand $\|w_\lambda\|_{\mathcal{W}^0([a,b])}^2 \leq \frac{1}{2\lambda}\|\zeta_\lambda\|_{\mathcal{W}^0([a,b])}^2$, thus it follows from (3.27) that

$$\sum_\lambda \lambda^4 \|w_\lambda\|_{\mathcal{W}^0([a,b])}^2 < +\infty. \quad (3.30)$$

From (3.29) and (3.30) follows that ξ satisfies (3.28). This proves *i*).

Step 4. Proof of *ii*. The proof of part *ii*) is analogous to the proof of part *i*). We just explain the differences. Let for example $a = 0$ and $b = +\infty$. In this case in order to prove the inequality (3.16) we repeat the same procedure as in the proof of the inequality (3.13) just with $b = +\infty$. Notice that in this case

$$\int_0^{+\infty} \partial_s \langle \xi, A\xi \rangle_{H^0} = -\langle \xi(0), A\xi(0) \rangle = -\|\pi^+(\xi(0))\|_{1/2} + \|\pi^-(\xi(0))\|_{1/2}.$$

The proof of the surjectivity of the maps F^\pm is also analog to the proof of surjectivity of the mapping F , thus we use again eigenspace decomposition of the function ξ, ζ and η . The solution ξ of the boundary value problem

$$\partial_s \xi + A\xi = \eta, \pi^+(\xi(0)) = \zeta^+$$

is given analogously to the equation (3.26). The eigenvectors $\xi_\lambda, \lambda > 0$ are given as in (3.26) in the case $a = 0$, whereas $\xi_\lambda, \lambda < 0$ are given by

$$\xi_\lambda(s, t) = - \int_s^{+\infty} \eta_\lambda(y, t) e^{\lambda(y-s)} dy.$$

The rest of the proof is word by word the same. \square

3.2 Elliptic regularity

In this section we shall prove some corollaries of the Theorem 3.1.6 for the specific choice of the linear operator A and its domain.

3.2.1 (The time independent case). Let $H^1 = H_{bc}^1([0, 1])$ and $H^0 = L^2([0, 1], \mathbb{R}^{2n})$ be as in (3.2) and suppose that the operator $A : H^1 \rightarrow H^0$ has the following form

$$A = J_0 \partial_t + S(t) : H_{bc}^1([0, 1]) \rightarrow L^2([0, 1]) \quad (3.31)$$

where J_0 is the standard complex structure. We assume that the operator A is bijective and self-adjoint. Let $E^\pm \subset L^2([0, 1])$ be generated by positive and negative eigenvectors as in 3.1.3 corresponding to the above operator A . Let $I = [a, b]$ or $I = \mathbb{R}^\pm$ and let

$$\begin{aligned} H_{bc}^1(I \times [0, 1]) &:= \{\xi \in W^{1,2}(I \times [0, 1]) \mid \xi(s, i) \in \mathbb{R}^n \times \{0\}\} \\ &= W^{1,2}(I, H^0) \cap L^2(I, H^1) \end{aligned} \quad (3.32)$$

Let $\mathcal{W}_\mp^1(I)$ be its subspace defined as follows

$$\begin{aligned} \mathcal{W}_\mp^1(I) &:= \{\xi \in H_{bc}^1(I \times [0, 1]) \mid \xi(a) \in E^-, \xi(b) \in E^+\} \\ &= \{\xi \in W^{1,2}(I, H^0) \cap L^2(I, H^1) \mid \xi(a) \in E^-, \xi(b) \in E^+\} \end{aligned} \quad (3.33)$$

Analogously in the case $I = \mathbb{R}^\pm$ we can define $\mathcal{W}_\mp^1(\mathbb{R}^\pm)$ as

$$\mathcal{W}_\mp^1(\mathbb{R}^\pm) := \{\xi \in H_{bc}^1(\mathbb{R}^\pm \times [0, 1]) \mid \xi(0) \in E^\mp\}.$$

Let

$$\begin{aligned} H_{bc}^2(I \times [0, 1]) &= \left\{ \xi \in W^{2,2}(I \times [0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, \ i = 0, 1 \\ A\xi(s, i) \in \mathbb{R}^n \times \{0\}, \ i = 0, 1 \end{array} \right\} \\ &= W^{2,2}(I, H^0) \cap W^{1,2}(I, H^1) \cap L^2(I, \text{Dom}(A^2)) \end{aligned} \quad (3.34)$$

Analogously we define a Hilbert subspace $\mathcal{W}_{\mp}^2(I)$ as follows

$$\mathcal{W}_{\mp}^2(I) := \{\xi \in H_{bc}^2(I \times [0, 1]) \mid \xi(a) \in E^-, \xi(b) \in E^+\}, \quad (3.35)$$

and in a similar way we define in the case $I = \mathbb{R}^{\pm}$ the space $\mathcal{W}_{\mp}^2(I)$. From Theorem 3.1.6 we derive some useful corollaries. The next corollary follows directly from the mentioned theorem.

Corollary 3.2.2 (Bijective linearized operator). *Let $I = [a, b]$ or $I = \mathbb{R}^{\pm}$ and let A be as in (3.31). Let $\mathcal{W}_{\mp}^i(I), i = 1, 2$ be defined as above. Denote with D_A the following linear operator*

$$\begin{aligned} D_A : \mathcal{W}_{\mp}^1(I) &\rightarrow L^2(I \times [0, 1]) \\ D_A \xi &= \partial_s \xi + A\xi. \end{aligned}$$

The operator D_A is bijective and similarly

$$D_A : \mathcal{W}_{\mp}^2(I) \rightarrow H_{bc}^1(I \times [0, 1])$$

is bijective.

Theorem 3.2.3. *Let $I = [a, b]$. Then the following hold*

i) (**$W^{1,2}$ regularity**) *Let $\xi, \eta \in L^2(I \times [0, 1])$ satisfy the following equality*

$$\int_0^1 \langle \xi, \cdot, D_A^* \zeta \rangle_{H^0} ds = \int_0^1 \langle \eta, \zeta \rangle_{H^0} ds, \quad \forall \zeta \in \mathcal{W}'(I), \quad (3.36)$$

where

$$\mathcal{W}'(I) = \{\xi \in L^2(I, H^1) \cap W^{1,2}(I, H^0) \mid \xi(a) \in E^+, \xi(b) \in E^-\}$$

and $D_A^ \zeta = -\partial_s \zeta + A\zeta$. Then $\xi \in \mathcal{W}_{\mp}^1(I)$ and $D_A \xi = \eta$.*

ii) (**$W^{2,2}$ regularity**) *Let $\xi \in \mathcal{W}_{\mp}^1(I)$ satisfy $D_A \xi = \eta$ for some $\eta \in H_{bc}^1(I \times [0, 1])$. Then $\xi \in \mathcal{W}_{\mp}^2(I)$.*

iii) (**$W^{1,p}$ regularity, $p > 2$**) *Let $\xi \in \mathcal{W}_{\mp}^1(I)$ be such that $D_A \xi = \eta \in L^p(I \times [0, 1])$. Then $\xi \in W_{bc}^{1,p}(I \times [0, 1]) := \{\xi \in W^{1,p}(I \times [0, 1]) \mid \xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1\}$.*

Proof. i) Let $\tilde{\xi} \in \mathcal{W}_+^1(I)$ be a unique solution of the equation $D_A \tilde{\xi} = \eta$. From Corollary 3.2.2 we have that such $\tilde{\xi}$ exists and is unique. Notice that for all $\zeta \in \mathcal{W}'(I)$ we have

$$\int_0^1 \langle \tilde{\xi}, D_A^* \zeta \rangle_{H^0} ds = \int_0^1 \langle D_A \tilde{\xi}, \zeta \rangle_{H^0} ds = \int_0^1 \langle \eta, \zeta \rangle_{H^0} ds.$$

Thus we have that

$$\int_0^1 \langle \tilde{\xi} - \xi, D_A^* \zeta \rangle_{H^0} ds = 0 \quad \forall \zeta \in \mathcal{W}'(I).$$

One can prove analogously as in Corollary 3.2.2 that the operator $D_A^* : \mathcal{W}'(I) \rightarrow L^2(I \times [0, 1])$ is bijective, thus we have that $\xi = \tilde{\xi}$.

ii) Let $\tilde{\xi} \in \mathcal{W}_+^2(I)$ be a unique solution of the equation $D_A \tilde{\xi} = \eta$. From Corollary 3.2.2 it follows that such $\tilde{\xi}$ exists and is unique. Notice that the difference $\xi' = \tilde{\xi} - \xi$ satisfies the equation $D_A \xi' = 0$ and $\xi' \in \mathcal{W}_+^1(I)$. Thus, it follows from Corollary 3.2.2 that $\tilde{\xi} = \xi$.

iii) Let $\tilde{\eta}$ be the extension of η on the whole of $\mathbb{R} \times [0, 1]$.

$$\tilde{\eta} = \begin{cases} \eta(s, t), & s \in I \\ 0, & s \notin I \end{cases}$$

It follows from Lemma 3.4.9 that the linear operator

$$D_A : W_{bc}^{1,p}(\mathbb{R} \times [0, 1]) \rightarrow L^p(\mathbb{R} \times [0, 1])$$

is bijective. Thus there exists a unique $\tilde{\xi} \in W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$ such that $D_A \tilde{\xi} = \tilde{\eta}$. Now in the case that $I = [a, b]$ for example it follows using eigenvector decomposition that $\tilde{\xi}(a) \in E^-$, as $D_A \tilde{\xi} = 0$ on the interval $(-\infty, a] \times [0, 1]$ and analogously $\tilde{\xi}(b) \in E^+$. As this implies that $\tilde{\xi} \in \mathcal{W}_+^1(I)$ and $D_A(\tilde{\xi} - \xi) = 0$, we have from Corollary 3.2.2 that $\tilde{\xi} = \xi \in W_{bc}^{1,p}(I \times [0, 1])$. \square

3.2.4 (The time dependent case). In the previous two sections we have examined linearized operators of the form $D_A = \partial_s + A$, where the operator A was bijective, self-adjoint and time independent. Now we allow the operator A to depend on time- s as well, but for simplicity we assume that the operators $A(s)$ have the following form

$$A(s) = J_0 \partial_t + S(s, t),$$

where J_0 is the standard complex structure and $S \in W^{1,2}(I \times [0, 1], \mathbb{R}^{2n \times 2n})$. Let $H^0 = L^2([0, 1])$ and suppose that $H^1 := \text{Dom}(A(s))$ and $H^2 := \text{Dom}(A(s)^2)$ are s independent. In other words the operators $A(s)$ can be written as the sum of some time independent operator A which satisfies (HA) and some matrix valued function R of $W^{1,2}$ class. Thus,

$$D\xi = \partial_s \xi + A(s)\xi = \partial_s \xi + A\xi + R(s, t)\xi,$$

where $R \in W^{1,2}(I \times [0, 1], \mathbb{R}^{2n \times 2n})$. Let I be either an interval of the form $I = [a, b]$ or $I = \mathbb{R}^\pm$ and let $H_{bc}^i(I \times [0, 1])$, $i = 1, 2$ be as in (3.32) and (3.34) respectively and let $H_{bc}^0(I \times [0, 1])$ be just the standard $L^2(I \times [0, 1])$. Observe the linear operator

$$\begin{aligned} D : H_{bc}^i(I \times [0, 1]) &\rightarrow H_{bc}^{i-1}(I \times [0, 1]), \quad i = 1, 2 \\ D\xi &= \partial_s \xi + J_0 \partial_t \xi + S(s, t)\xi = \partial_s \xi + A(s)\xi = \partial_s \xi + A\xi + R(s, \cdot)\xi \end{aligned} \quad (3.37)$$

(H1) In the case $I = \mathbb{R}^\pm$ we additionally assume that the

$$\lim_{s \rightarrow \pm\infty} \|S(s, t) - S^\pm(t)\|_{C^1([s, \infty) \times [0, 1])} = 0,$$

where $S^\pm : [0, 1] \rightarrow \mathbb{R}^{2n \times 2n}$ are smooth functions and that the limit operators

$$A^\pm = J_0 \partial_t + S^\pm(t) : H^1 \rightarrow H^0$$

satisfy (HA).

The analogous statement as in Theorem 3.2.3 holds in the case of time-dependent operator A . We formulate and prove the analogous theorem.

Theorem 3.2.5 (Regularity). *Let $I = [a, b]$ and let the linear operator D be as in (3.37). Then the following statements hold*

i) *Let $\xi, \eta \in L^2(I \times [0, 1])$ and suppose that the following equality*

$$\int_I \langle \xi, D^* \zeta \rangle_{H^0} ds = \int_I \langle \eta, \zeta \rangle_{H^0} ds,$$

holds for all $\xi \in \mathcal{W}'(I) = \{\xi \in L^2(I, H^1) \cap W^{1,2}(I, H^0) \mid \zeta(a) \in E^+, \zeta(b) \in E^-\}$, where $D^ = D_A^* + R^T = -\partial_s + A + R^T$. Then ξ is a strong solution of the equation $D\xi = \eta$ and $\xi \in \mathcal{W}_\mp^1(I)$.*

ii) *Let $\xi \in \mathcal{W}_\mp^1(I)$, $\eta \in H_{bc}^1(I \times [0, 1])$ satisfy $D\xi = \eta$. Then $\xi \in \mathcal{W}_\mp^2(I)$.*

iii) *Let $\xi \in \mathcal{W}_\mp^1(I)$ and $\eta \in L^p(I \times [0, 1])$, $p > 2$ satisfy the equation $D\xi = \eta$. Then $\xi \in W_{bc}^{1,p}(I \times [0, 1])$.*

Proof. i) Notice that

$$\begin{aligned} \int_I \langle \xi, D^* \zeta \rangle_{H^0} ds &= \int_I \langle \xi, D_A^* \zeta + R^T \zeta \rangle_{H^0} ds \\ &= \int_I \langle \xi, D_A^* \zeta \rangle_{H^0} ds + \int_I \langle R \xi, \zeta \rangle_{H^0} ds. \end{aligned}$$

Thus we have that

$$\int_I \langle \xi, D_A^* \zeta \rangle_{H^0} ds = \int_I \langle \eta - R \xi, \zeta \rangle_{H^0} ds = \int_I \langle \eta', \zeta \rangle_{H^0} ds,$$

and $\eta' = \eta - R \xi \in L^2(I \times [0, 1])$. Thus it follows from Theorem 3.2.3, that $\xi \in \mathcal{W}_\mp^1(I)$ and it satisfies the equality $D_A \xi = \eta'$. Thus $D(\xi) = D_A \xi + R \xi = \eta$.

ii) As $D \xi = D_A \xi + R \xi = \eta$, and both ξ and R are $W^{1,2}$ functions their product will be an L^p function for any $p > 1$. Thus, $D_A \xi = \eta - R \xi \in L^p(I \times [0, 1])$, $p > 2$. From Theorem 3.2.3 it follows that $\xi \in W_{bc}^{1,p}(I \times [0, 1])$. This implies that the product $R \cdot \xi \in W_{bc}^{1,2}(I \times [0, 1])$ and $D_A \xi = \eta' \in W_{bc}^{1,2}(I \times [0, 1])$. Again, from Theorem 3.2.3 we have that $\xi \in \mathcal{W}_\mp^2(I)$. The proof of part iii) is analogous to the proof of ii) and we shall not repeat it. \square

Theorem 3.2.6 (Estimates). *Let H^i , $i = 0, 1, 2$ be as in 3.2.4 and let D be as in (3.37). Let $A = J_0 \partial_t + S_1(t) : H^1 \rightarrow H^0$ and $B = J_0 \partial_t + S_2(t) : H^1 \rightarrow H^0$ satisfy (HA). Let E_A^\pm, E_B^\pm be Hilbert spaces generated by positive and negative eigenvectors of the operators A and B respectively as in 3.1.3. Denote with π_A^\pm and π_B^\pm the corresponding projections, as in 3.1.3.*

i) Let $I = [a, b]$. Then there exist a constant $c > 0$ and a compact operator

$$K : H_{bc}^2(I \times [0, 1]) \rightarrow H_{bc}^1(I \times [0, 1])$$

such that the following inequality holds for all $\xi \in H_{bc}^2(I \times [0, 1])$.

$$\|\xi\|_{2,2} \leq c \left(\|D \xi\|_{1,2} + \|K \xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_B^-(\xi(b))\|_{3/2} \right). \quad (3.38)$$

ii) Let $I = \mathbb{R}^\pm$ and assume (H1). Then there exist a constant $c > 0$ and a compact operator $K : H_{bc}^2(\mathbb{R}^\pm \times [0, 1]) \rightarrow H_{bc}^1(\mathbb{R}^\pm \times [0, 1])$ such that the following inequality holds for all $\xi \in H_{bc}^2(\mathbb{R}^\pm \times [0, 1])$.

$$\|\xi\|_{2,2} \leq c \left(\|D \xi\|_{1,2} + \|K \xi\|_{1,2} + \|\pi_A^\pm(\xi(0))\|_{3/2} \right). \quad (3.39)$$

Proof. We prove this theorem in the next four steps.

Step 1. Proof of the inequality (3.38) in the case that $A = B$.

Denote with D_A linear operator $D_A = \partial_s + A = J_0 \partial_t + S_1(t)$. From Theorem 3.1.6 we have the following:

$$\begin{aligned} \|\xi\|_{2,2} &\leq c \left(\|D_A \xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_A^-(\xi(b))\|_{3/2} \right) \\ &\leq c \left(\|D\xi\|_{1,2} + \|(D - D_A)\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_A^-(\xi(b))\|_{3/2} \right) \\ &\leq c \left(\|D\xi\|_{1,2} + \|(S - S_1)\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_A^-(\xi(b))\|_{3/2} \right) \\ &\leq c \left(\|D\xi\|_{1,2} + \|R\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_A^-(\xi(b))\|_{3/2} \right). \end{aligned} \quad (3.40)$$

Notice that the difference $S(s, t) - S_1(t) = R(s, t)$ and the matrix valued function $R \in W^{1,2}(I \times [0, 1])$. The operator $K(\xi) := R\xi$ is a compact operator. This follows by the following observation

$$\begin{aligned} \|K(\xi)\|_{1,2} &= \|R\xi\|_{1,2} \leq \|dR\xi\|_{L^2} + \|Rd\xi\|_{L^2} + \|R\xi\|_{L^2} \\ &\leq \|dR\|_{L^2} \|\xi\|_{L^\infty} + \|R\|_{L^4} \|d\xi\|_{L^4} + \|R\|_{L^4} \|\xi\|_{L^4} \\ &\leq \|R\|_{1,2} \|\xi\|_{L^\infty} + c \|R\|_{1,2} \|\xi\|_{1,4} + c \|R\|_{1,2} \|\xi\|_{L^4} \end{aligned}$$

As the embedding $W^{2,2}(I \times [0, 1]) \hookrightarrow W^{1,4}(I \times [0, 1])$ is compact as well as the embedding $W^{2,2}(I \times [0, 1]) \hookrightarrow L^\infty$, we have that the operator K is compact.

Step 2. Proof of the inequality (3.39) in the case that $A^\pm = A$.

We do the proof in the case of positive half-infinite strips. The case of negative strips is analogous. Consider the following linear maps

$$\begin{aligned} F, F_A : H_{bc}^2(\mathbb{R}^+ \times [0, 1], \mathbb{R}^{2n}) &\rightarrow H_{bc}^1(\mathbb{R}^+ \times [0, 1]) \times E_A^+, \\ F(\xi) &= (D\xi, \pi^+(\xi(0))), \quad F_A(\xi) = (D_A \xi, \pi^+(\xi(0))). \end{aligned}$$

From Theorem 3.1.6 it follows that the map F_A is bijective and it satisfies the estimate:

$$\|\xi\|_{2,2} \leq c \left(\|D_A(\xi)\|_{1,2} + \|\pi_A^+(\xi(0))\|_{3/2} \right). \quad (3.41)$$

The operator F is just a compact perturbation of the operator F_A what can be proved as follows. Convergence $S(s, t) \xrightarrow{C^1} S_1(t)$ implies that for s_0 sufficiently large we have

$$\|S(s, t) - S_1(t)\|_{C^1([s_0, +\infty) \times [0, 1])} \leq \frac{1}{4c},$$

where c is the constant of the inequality (3.41). Let $\beta : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function with

$$\beta(s) = \begin{cases} 1, & s \leq s_0, \\ 0, & s \geq s_1 \gg s_0 \end{cases}$$

and $\|\beta\|_{C^1} \leq 2$. From (3.41) we obtain

$$\begin{aligned} \|\xi\|_{2,2} &\leq c(\|D\xi\|_{1,2} + \|(D - D_A)\xi\|_{1,2} + \|\pi_A^+(\xi(0))\|_{3/2}) \\ &\leq c(\|D\xi\|_{1,2} + \|(S(s, t) - S_1(t))\xi\|_{1,2} + \|\pi_A^+(\xi(0))\|_{3/2}) \\ &\leq c(\|D\xi\|_{1,2} + \|(S(s, t) - S_1(t))\beta\xi\|_{1,2} \\ &\quad + \|(S(s, t) - S_1(t))(1 - \beta)\xi\|_{1,2} + \|\pi^+(\xi(0))\|_{3/2}) \\ &\leq 2c(\|D\xi\|_{1,2} + \|K\xi\|_{W^{1,2}([0, s_1] \times [0, 1])} + \|\pi_A^+(\xi(0))\|_{3/2}). \end{aligned} \quad (3.42)$$

Here the operator K is given as multiplication by $(S - S_1)\beta$ which has compact support and is $W^{1,2}$ function.

Step 3. Proof of (3.38) in general case.

In Step 1) we have proved the following inequality

$$\|\xi\|_{2,2} \leq c(\|D\xi\|_{2,2} + \|K\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_A^-(\xi(b))\|_{3/2}) \quad (3.43)$$

and the same inequality follows when the operator A is substituted with the operator B on the right side of the inequality. Let $\beta : [a, b] \rightarrow [0, 1]$ be a smooth cut-off function such that

$$\beta(s) = \begin{cases} 1, & s \leq a + \frac{b-a}{4}, \\ 0, & s \geq b - \frac{b-a}{4} \end{cases}$$

Apply the inequality (3.43) to $\beta\xi$, and the same type of the inequality just with A substituted with B to $(1 - \beta)\xi$. Thus the following two inequalities hold

$$\|\beta\xi\|_{2,2} \leq c(\|D(\beta\xi)\|_{1,2} + \|\beta K\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2}) \quad (3.44)$$

and analogously we have

$$\|(1 - \beta)\xi\|_{2,2} \leq c(\|D((1 - \beta)\xi)\|_{1,2} + \|(1 - \beta)K\xi\|_{1,2} + \|\pi_B^-(\xi(b))\|_{3/2}). \quad (3.45)$$

Summing the inequalities (3.44) and (3.45) we obtain

$$\begin{aligned}
 \|\xi\| &\leq c \left(\|\beta D\xi\|_{1,2} + \|(1-\beta)D\xi\|_{1,2} + \|\dot{\beta}\xi\|_{1,2} + \|K\xi\|_{1,2} \right. \\
 &\quad \left. + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_B^-(\xi(b))\|_{3/2} \right) \\
 &\leq c \left(\|D\xi\|_{1,2} + \|\xi\|_{1,2} + \|K\xi\|_{1,2} + \|\pi_A^+(\xi(a))\|_{3/2} + \|\pi_B^-(\xi(b))\|_{3/2} \right)
 \end{aligned} \tag{3.46}$$

As, also the embedding $H_{bc}^2(I \times [0, 1]) \rightarrow H_{bc}^1(I \times [0, 1])$ is compact, the claim follows.

Step 4: Proof of (3.39) in general.

We prove the inequality in the case of positive half strips. The proof in the case of negative strips is analogous. Let $\beta : \mathbb{R}^+ \rightarrow [0, 1]$ be a smooth cut-off function with the properties:

$$\beta(s) = \begin{cases} 1, & s \geq s_1 \\ 0, & s \leq s_0 \end{cases}$$

Applying the results of Step 2) to $\beta\xi$ and the limit operator A^+ we obtain

$$\begin{aligned}
 \|\beta\xi\|_{2,2} &\leq c \left(\|D(\beta\xi)\|_{1,2} + \|\beta K\xi\|_{1,2} + \|\pi_{A^+}^+(\beta\xi(0))\|_{3/2} \right) \\
 &\leq c \left(\|\beta D\xi\|_{1,2} + \|\dot{\beta}\xi\|_{1,2} + \|K\xi\|_{1,2} \right) \\
 &\leq c \left(\|D\xi\|_{1,2} + \|K'\xi\|_{1,2} \right)
 \end{aligned} \tag{3.47}$$

where K' is compact operator. Apply next the inequality (3.38) to $(1-\beta)\xi$ and the operator A on compact interval $[0, s_1] \times [0, 1]$. We have

$$\begin{aligned}
 \|(1-\beta)\xi\|_{2,2} &\leq c \left(\|D((1-\beta)\xi)\|_{1,2} + \|(1-\beta)K\xi\|_{1,2} + \|\pi_A^+(\xi(0))\|_{3/2} \right) \\
 &\leq c \left(\|(1-\beta)D\xi\|_{1,2} + \|K\xi\|_{1,2} + \|(1-\beta)\xi\|_{1,2} \|\pi_A^+(\xi(0))\|_{3/2} \right) \\
 &\leq c \left(\|D\xi\|_{1,2} + \|K'\xi\|_{1,2} + \|\pi_A^+(\xi(0))\|_{3/2} \right)
 \end{aligned} \tag{3.48}$$

Summing the inequalities (3.47) and (3.48) we obtain the inequality (3.39). \square

3.2.7 (Closed Image). Let D be the operator as in (3.37). We prove that D has closed image.

Lemma 3.2.8. *Let $i = 1$ or $i = 2$ and let $I = [a, b]$ or $I = \mathbb{R}^\pm$. Let D be as in (3.37). In the case of infinite strips $I = \mathbb{R}^\pm$ we assume (H1). Then the image of the operator D is closed.*

Proof. We prove Lemma 3.2.8 in the case $I = [a, b]$ and $i = 2$. The proof in the case $I = \mathbb{R}^\pm$ and in the case $i = 1$ is analogous. Let $A = J_0 \partial_t + S(t) : H^1 \rightarrow H^0$ and E^\pm be as in 3.2.1 and let $\mathcal{V} = \mathcal{W}_\pm^2(I)$ be defined as in (3.35).

Step 1. The restriction of the operator D to \mathcal{V} is a Fredholm operator.

Let $D_A = \partial_s + A$. Then it follows from the corollary 3.2.2 that the restriction of operator D_A to \mathcal{V} is a bijective operator. Thus particularly the restriction of the operator D_A to \mathcal{V} is a Fredholm operator of index 0. On the other hand the operator $D = D_A + (S(s, t) - S_1(t)) = D_A + K$, where $K : H_{bc}^2(I \times [0, 1]) \rightarrow H_{bc}^1(I \times [0, 1])$ is a compact operator. Thus the operator D is a compact perturbation of a Fredholm operator, hence it is also a compact operator of the same index.

Step 2. The operator $D : H_{bc}^2(I \times [0, 1]) \rightarrow H_{bc}^1(I \times [0, 1])$ has closed image.

Let $X = H_{bc}^2(I \times [0, 1])$, $Y = H_{bc}^1(I \times [0, 1])$ and let $\mathcal{V} \subset X$ be as in Step 1.

Denote with Y_0 the image of \mathcal{V} via D , i.e. $Y_0 = D(\mathcal{V})$. Then it follows from Step 1 that $Y_0 \subset Y$ is closed and finite codimension subspace. We need to prove that $Y_1 = D(X)$ is also closed. Notice that $Y_0 \subset Y_1 \subset Y$. Observe natural projection $\text{pr} : Y_1 \rightarrow Y/Y_0$. As Y/Y_0 is finite dimensional space and $\text{pr}(Y_1)$ is a vector subspace it follows that $\text{pr}(Y_1)$ is finite dimensional and hence also closed in Y/Y_0 . Thus $Y_1 = \text{pr}^{-1}(\text{pr}(Y_1))$ is closed in Y . \square

We prove in Section 3.3 that D is actually surjective.

3.3 Unique continuation and surjectivity

3.3.1 (Elliptic regularity at the corner). Here we shall prove elliptic regularity at the corner which is reduced using reflection argument to the elliptic regularity at the boundary. As a corollary we prove that the operator D as in (3.37) has dense image and as a corollary we prove that it is also surjective.

Let $\epsilon \in \overline{R}$ be positive and define

$$\Omega = [0, \epsilon) \times [0, 1], \quad \tilde{\Omega} = (-\epsilon, \epsilon) \times [0, 1] \quad (3.49)$$

Denote by $C_{c,bc}^\infty(\tilde{\Omega})$ the following set

$$C_{c,bc}^\infty(\tilde{\Omega}) = \left\{ \phi \in C_c^\infty(\tilde{\Omega}, \mathbb{R}^{2n}) \mid \phi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \right\} \quad (3.50)$$

Similarly we define the set $C_{c,bc}^\infty(\Omega)$ by

$$C_{c,bc}^\infty(\Omega) = \{ \phi \in C_c^\infty(\Omega, \mathbb{R}^{2n}) \mid \phi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \}$$

Notice that a function $\phi \in C_{c,bc}^\infty(\Omega)$ doesn't vanish on $\{0\} \times [0, 1]$.

A direct corollary of Lemma B.4.9 in [16] is the following:

Claim 3.3.2. *If a function $u \in L_{loc}^2(\tilde{\Omega}, \mathbb{R}^{2n})$ satisfies the following equality*

$$\int_{\tilde{\Omega}} \langle \partial_s \phi + J_0 \partial_t \phi, u \rangle = \int_{\tilde{\Omega}} \langle \phi, v \rangle, \quad \forall \phi \in C_{c,bc}^\infty(\tilde{\Omega}, \mathbb{R}^{2n}) \quad (3.51)$$

where $v \in L_{loc}^2(\tilde{\Omega})$. Then the following holds

- 1) $u \in H_{loc}^1(\tilde{\Omega}, \mathbb{R}^{2n})$ and $u(s, i) \in \mathbb{R}^n \times \{0\}$ for $i = 0, 1$.
- 2) $-\partial_s u + J_0 \partial_t u = v$

Lemma 3.3.3. *Let Ω be as in (3.49). Suppose that $\eta \in L_{loc}^2(\Omega)$ satisfies*

$$\int_{\Omega} \langle \partial_s \phi + J_0 \partial_t \phi, \eta \rangle = \int_{\Omega} -\langle \zeta, \phi \rangle \quad (3.52)$$

for all $\phi \in C_{c,bc}^\infty(\Omega)$ and some $\zeta \in L_{loc}^2(\Omega)$. Then the following holds

- 1) $\eta \in H_{loc}^1(\Omega)$, $\eta(s, 0), \eta(s, 1) \in \mathbb{R}^n \times 0$ and $\eta(0, t) = 0$, for a.e. $t \in [0, 1]$.
- 2) $-\partial_s \eta + J_0 \partial_t \eta = -\zeta$.

Proof of Lemma 3.3.3. The proof follows from the Claim 3.3.2 and a reflection argument. Let η and ζ satisfy the equation (3.52). We shall extend both η and ζ to $\tilde{\Omega}$ in two different ways.

I) Let $\tilde{\eta}, \tilde{\zeta}$ be odd and even extensions of η and ζ .

$$\tilde{\eta}(s, t) = \begin{cases} \eta(s, t), & s \geq 0 \\ -\overline{\eta(-s, t)}, & s < 0 \end{cases} \quad \tilde{\zeta}(s, t) = \begin{cases} \zeta(s, t), & s \geq 0 \\ \overline{\zeta(-s, t)}, & s < 0 \end{cases}$$

Here $\overline{\zeta}$ represents the image of ζ made by symmetry with respect to the plane $x_1 = x_2 = \dots = x_n = 0$, or equivalently

$$\overline{\zeta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \zeta$$

We shall prove that the extended functions $\tilde{\eta}$ and $\tilde{\zeta}$ satisfy the equation (3.51), i.e. for all $\phi \in C_{c,bc}^\infty(\tilde{\Omega})$ the following holds:

$$\int_{\tilde{\Omega}} \langle \partial_s \phi + J_0 \partial_t \phi, \tilde{\eta} \rangle = - \int_{\tilde{\Omega}} \langle \phi, \tilde{\zeta} \rangle \quad (3.53)$$

Let $\phi \in C_{c,bc}^\infty(\tilde{\Omega})$ be an arbitrary function. Define functions ϕ_0 and ϕ_1 in the following way

$$\phi_0(s, t) = \frac{1}{2} \left(\phi(s, t) + \overline{\phi(-s, t)} \right), \quad \phi_1(s, t) = \frac{1}{2} \left(\phi(s, t) - \overline{\phi(-s, t)} \right)$$

Obviously $\phi_0(-s, t) = \overline{\phi_0(s, t)}$, $\phi_1(-s, t) = -\overline{\phi_1(s, t)}$ and $\phi = \phi_0 + \phi_1$. It also holds

$$\partial_s \phi_0(-s, t) = -\overline{\partial_s \phi_0(s, t)}, \quad \partial_t \phi_0(-s, t) = \overline{\partial_t \phi_0(s, t)}.$$

Hence, we have that $\partial_s \phi_0(-s, t) + J_0 \partial_t \phi_0(-s, t) = -(\overline{\partial_s \phi_0(s, t) + J_0 \partial_t \phi_0(s, t)})$. For ϕ_1 the reverse holds i.e.

$$\partial_s \phi_1(-s, t) + J_0 \partial_t \phi_1(-s, t) = \overline{\partial_s \phi_1(s, t) + J_0 \partial_t \phi_1(s, t)}.$$

Now it is easy to see that

$$\int_{\tilde{\Omega}} \langle \partial_s \phi_1 + J_0 \partial_t \phi_1, \tilde{\eta} \rangle = 0 = \int_{\tilde{\Omega}} \langle \phi_1, \tilde{\zeta} \rangle$$

and also

$$\begin{aligned} \int_{\tilde{\Omega}} \langle \partial_s \phi_0 + J_0 \partial_t \phi_0, \tilde{\eta} \rangle &= 2 \int_{\tilde{\Omega}} \langle \partial_s \phi_0 + J_0 \partial_t \phi_0, \eta \rangle \\ &\quad - \int_{\tilde{\Omega}} \langle \phi_0, \tilde{\zeta} \rangle = -2 \int_{\tilde{\Omega}} \langle \phi_0, \zeta \rangle \end{aligned} \quad (3.54)$$

By assumption (3.52) the integrals on the right side of (3.54) are equal, it follows

$$\int_{\tilde{\Omega}} \langle \partial_s \phi + J_0 \partial_t \phi, \tilde{\eta} \rangle = - \int_{\tilde{\Omega}} \langle \phi, \tilde{\zeta} \rangle. \quad (3.55)$$

Thus, we have proved the equation (3.53). From corollary 3.3.2 we have that $\tilde{\eta} \in H_{bc}^1(\tilde{\Omega}, \mathbb{R}^{2n})$ as $\eta = \tilde{\eta}|_{\Omega}$ we have that η also satisfies

$$\eta(s, i) \in \mathbb{R}^n \times \{0\}, \eta \in H_{loc}^1(\Omega).$$

We prove that $\tilde{\eta}(0, t) = \eta(0, t) \in 0 \times \mathbb{R}^n$ for almost every t . Choose a sequence of smooth functions $\eta_i(s, t)$ that converges to $\tilde{\eta}$ on every compact subset of $\tilde{\Omega}$ in $W^{1,2}$ norm. Notice that $\tilde{\eta} = \frac{1}{2}(\tilde{\eta}(s, t) - \overline{\tilde{\eta}(-s, t)})$. Then the sequence $h_i(s, t) = \frac{1}{2}(\eta_i(s, t) - \overline{\eta_i(-s, t)})$ also converges to $\tilde{\eta}$ in $W^{1,2}$ norm. We also have that $h_i(0, t) = \frac{1}{2}(\eta_i(0, t) - \overline{\eta_i(0, t)}) \in \{0\} \times \mathbb{R}^n$. As $h_i(0, t) \xrightarrow{L^2} \eta(0, t) = \tilde{\eta}(0, t)$ it follows that $\eta(0, t) \in \{0\} \times \mathbb{R}^n$ for almost every t .

II) We will extend now η and ζ reverse than in the case **I**), i.e. η even and ζ odd. Define $\tilde{\eta}$ and $\tilde{\zeta}$ as

$$\tilde{\eta}(s, t) = \begin{cases} \eta(s, t), & \text{if } s \geq 0 \\ \overline{\eta(-s, t)}, & \text{if } s < 0 \end{cases} \quad \tilde{\zeta}(s, t) = \begin{cases} \zeta(s, t), & \text{if } s \geq 0 \\ -\overline{\zeta(-s, t)}, & \text{if } s < 0. \end{cases}$$

Extended functions $\tilde{\eta}$ and $\tilde{\zeta}$ satisfy (3.53). In order to prove that one can use the same decomposition of function ϕ . This time the integral with ϕ_1 will be doubled and the integral with ϕ_0 will vanish. Therefore we conclude from the Claim 3.3.2 that $\tilde{\eta} \in W_{loc}^{1,2}(\tilde{\Omega})$. In the same way as in **(I)** we prove that

$$\tilde{\eta}(0, t) = \eta(0, t) \in \mathbb{R}^n \times \{0\} \text{ for a.e. } t.$$

Thus from **I)** and from what we have just proved

$$\eta(0, t) \in (\mathbb{R}^n \times \{0\}) \cap (\{0\} \times \mathbb{R}^n) = 0$$

for almost every $t \in [0, 1]$. □

3.3.4 (Unique continuation). Let $I = [0, \epsilon)$, where ϵ is possibly infinite and let D be an operator as in (3.37). In this paragraph we observe the mapping

$$\xi \mapsto (D\xi, \xi(0)) = (\partial_s \xi + J_0 \partial_t \xi + S\xi, \xi(0))$$

In the case $\epsilon = +\infty$ we suppose (H1). This mapping is injective, i.e. any ξ with $\xi(0, \cdot) = 0$ and $D\xi = 0$ has to vanish everywhere $\xi \equiv 0$. This follows from Agmon and Nirenberg trick and we shall not discuss the details of the proof here. It is proved by careful study of the function $\ln(\|\xi\|)$ and can be seen in [21].

Lemma 3.3.5. *Let $i = 1$ or $i = 2$ and let D be an operator as in (3.37). Then the mapping*

$$\begin{aligned} H_{bc}^i([0, \epsilon) \times [0, 1]) &\rightarrow H_{bc}^{i-1}([0, \epsilon) \times [0, 1]) \times H_{bc}^{i-1/2} \\ \xi &\mapsto (D\xi, \xi(0)) \end{aligned} \quad (3.56)$$

is injective. The analog holds for the operator $D^ = -\partial_s + J_0\partial_t + S(s, t)^T$. Namely, the mapping*

$$\begin{aligned} H_{bc}^i([0, \epsilon) \times [0, 1]) &\rightarrow H_{bc}^{i-1}([0, \epsilon) \times [0, 1]) \times H_{bc}^{i-1/2} \\ \xi &\mapsto (D^*\xi, \xi(0)) \end{aligned} \quad (3.57)$$

is injective.

Proof. The proof is verbatim the same as the proof of Lemma 3.3 in [21] and we shall not repeat it here. \square

Corollary 3.3.6. *Let D be an operator of the form (3.37).*

i) Let $I = [a, b]$. There exists a constant $c > 0$ such that the following inequality holds for all $\xi \in H_{bc}^2(I \times [0, 1])$.

$$\|\xi\|_{2,2} \leq c (\|D\xi\|_{1,2} + \|\xi(a, \cdot)\|_{3/2} + \|\xi(b, \cdot)\|_{3/2}) \quad (3.58)$$

ii) Suppose that $I = \mathbb{R}^\pm$ and assume (H1). Then there exist positive constant c such that the following inequality holds for all $\xi \in H_{bc}^2(\mathbb{R}^\pm \times [0, 1])$.

$$\|\xi\|_{2,2} \leq c (\|D\xi\|_{1,2} + \|\xi(0, \cdot)\|_{3/2}) \quad (3.59)$$

Proof. We shall prove part ii), the proof of i) is analogous. It follows directly from the inequality (3.39) in Theorem 3.2.6 that

$$\|\xi\|_{2,2} \leq c (\|D\xi\|_{1,2} + \|K\xi\|_{1,2} + \|\xi(0)\|_{3/2}), \quad (3.60)$$

as $\|\pi^+(\xi(0))\|_{3/2} \leq \|\xi(0)\|_{3/2}$. Remember also that the operator $H_{bc}^2(\mathbb{R}^+ \times [0, 1]) \ni \xi \mapsto K\xi \in H_{bc}^1(\mathbb{R}^+ \times [0, 1])$ is compact. From the inequality (3.60) it follows that the operator $\xi \mapsto (D\xi, \xi(0))$ has closed image and finite dimensional kernel. By Lemma 3.3.5 it follows that it is bijective onto its image. From the open mapping theorem it follows that its inverse is bounded and we can omit the middle term, i.e. $\|K\xi\|_{1,2}$ of the inequality (3.60). Thus we have proved the required inequality. \square

Corollary 3.3.7. *Let $I = [a, b]$ or $I = \mathbb{R}^\pm$. Suppose that the operator $D : H_{bc}^i(I \times [0, 1]) \rightarrow H_{bc}^{i-1}(I \times [0, 1])$, $i = 1, 2$ has the form (3.37), in the case $I = \mathbb{R}^\pm$ we assume (H1). Then the operator D is surjective.*

Proof. **Step 1.** Surjectivity of the operator

$$D : H_{bc}^1(I \times [0, 1]) \rightarrow L^2(I \times [0, 1]).$$

Let $\Omega = I \times [0, 1]$ and let $\eta \in L^2(\Omega)$ be orthogonal to the image of D . Then we have that

$$\int_{\Omega} \langle \partial_s \xi + J_0 \partial_t \xi + S \xi, \eta \rangle = 0$$

holds for all $\xi \in H_{bc}^1(\Omega)$. Particularly this implies that for all $\phi \in C_{c,bc}^\infty(\Omega)$ (defined in (3.50)) the following equality holds

$$\int_I \int_0^1 \langle \partial_s \phi + J_0 \partial_t \phi, \eta \rangle ds dt = - \int_I \int_0^1 \langle S^T \eta, \phi \rangle = - \int_I \int_0^1 \langle \zeta, \phi \rangle ds dt.$$

It follows from Lemma 3.3.3 that $\eta \in H_{bc}^1(I \times [0, 1])$ and that $\eta|_{\partial I} = 0$ and η is a strong solution of the equation

$$-\partial_s \eta + J_0 \partial_t \eta = -\zeta = -S^T \eta.$$

From Lemma 3.3.5 it follows that $\eta = 0$. Thus the image of the operator

$$D : H_{bc}^1(\Omega) \rightarrow L^2(\Omega), \quad D\xi = \partial_s \xi + J_0 \partial_t \xi + S(s, t)\xi$$

is dense and as its image is also closed (see Lemma 3.2.8) we have that D is surjective.

Step 2. Let $\mathcal{W}_{\mp}^1(I)$ be defined as in (3.33). Then there exist smooth functions $\xi_i \in H_{bc}^1(I \times [0, 1])$, $i = 1, \dots, m$ such that

$$D : \mathcal{W}_{\mp}^1(I) \cup \text{Span}\{\xi_1, \dots, \xi_m\} \rightarrow L^2(I \times [0, 1])$$

is surjective.

Notice that the operator D can be written in the form $D = D_A + R$, where

$R(\xi)$ is given as a multiplication by some $W^{1,2}$ matrix valued function. As the operator $D_A : \mathcal{W}_{\mp}^1(I) \rightarrow L^2$ is bijective (Corollary 3.2.2), we have that the operator $D : \mathcal{W}_{\mp}^1(I) \rightarrow L^2$ is Fredholm of index 0, as a compact perturbation of the operator D_A . From Step 1 it follows that there exist ξ_i , $i = 1, \dots, m$ such that the restriction of the operator D to the $\mathcal{W}_{\mp}^1(I) \cup \text{Span}\{\xi_1, \dots, \xi_m\}$ is surjective. Notice that each ξ_i can be approximated by smooth elements ξ_i^k , $k = 1, \dots, \infty$ which also satisfy the condition $D_A \xi_i^k \in H_{bc}^1(I \times [0, 1])$.

Thus we have that $\xi_i^k \in H_{bc}^2(I \times [0, 1])$. Thus for sufficiently large k we have that the restriction of the operator D to $\mathcal{W}_\mp^1(I) \cup \text{Span}(\xi_1^k, \dots, \xi_m^k)$ is surjective. Thus, we can assume w.l.o.g. that $\xi_i \in H_{bc}^2(I \times [0, 1])$ are smooth.

Step 3. Let $\mathcal{W}_\mp^2(I)$ be defined as in (3.35) and let $\xi_i, i = 1, \dots, m$ be as in Step 2. Then

$$D : \mathcal{W}_\mp^2(I) \cup \text{Span}\{\xi_1, \dots, \xi_m\} \rightarrow H_{bc}^1(I \times [0, 1])$$

is surjective.

Let $\eta \in H_{bc}^1(I \times [0, 1])$. From Step 2 it follows that there exist $\xi \in \mathcal{W}_\mp^1(I)$ and $\alpha_i \in \mathbb{R}$ such that $D(\xi + \alpha_i \xi_i) = \eta$. We prove that $\xi \in \mathcal{W}_\mp^2(I)$ actually. First notice that

$$D\xi = \eta - \sum_i \alpha_i D\xi_i = \eta - \sum_i \eta_i = \eta' \in H_{bc}^1(I \times [0, 1])$$

thus we have that

$$D\xi = D_A\xi + R\xi = \eta' \quad \Rightarrow \quad D_A\xi = \eta' - R\xi = \tilde{\eta}$$

As R is a $W^{1,2}$ function and ξ as well, we have that their product is an L^p function for any $p < \infty$. Thus the function $\tilde{\eta} \in L^p(I \times [0, 1])$ and it follows from Theorem 3.2.5 that $\xi \in W_{bc}^{1,p}(I \times [0, 1])$ for some $p > 2$. This implies that the product $R\xi$ is actually a $W^{1,2}$ function and it also satisfies the right boundary condition and hence $\tilde{\eta} \in H_{bc}^1(I \times [0, 1])$. From Corollary 3.2.2 we have that $\xi \in \mathcal{W}_\mp^2(I)$.

Steps 1-3 prove that the operator D is surjective. \square

3.4 Appendix

3.4.1 Abstract interpolation theory

Let H and W be Hilbert spaces and let the inclusion $W \hookrightarrow H$ be continuous and dense. We define the space $\mathcal{W} = \mathcal{W}(0, +\infty)$ as follows

$$\begin{aligned} \mathcal{W} = \mathcal{W}(0, +\infty) &= \left\{ x : x \in L^2((0, +\infty), W), \frac{\partial u}{\partial s} \in L^2((0, +\infty), H) \right\} \\ &= \left\{ x \in L^2((0, +\infty), W) \cap W^{1,2}((0, +\infty), H) \right\} \end{aligned}$$

with the norm

$$\|x\|_{\mathcal{W}}^2 = \|x\|_{L^2((0, +\infty), W)}^2 + \|\dot{x}\|_{L^2((0, +\infty), H)}^2 = \int_0^{+\infty} (\|x(s)\|_W^2 + \|\dot{x}(s)\|_H^2) ds$$

Remark 3.4.1. The space \mathcal{W} is a Hilbert space and the space $C_c^\infty([0, +\infty), W)$ is dense in \mathcal{W} .

Definition 3.4.2. The trace space V of the space \mathcal{W} is given by

$$V = Tr(\mathcal{W}) := \left\{ \xi \in H : \exists x \in \mathcal{W}, x(0) = \xi \right\}$$

$$\|\xi\|_{1/2} = \|\xi\|_V^2 := \inf_{x \in \mathcal{W}, x(0) = \xi} \int_0^{+\infty} (\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2) ds$$

Remark 3.4.3. One could also use finite interval $I = (0, 1)$ or $I = \mathbb{R}$ instead of the interval $(0, +\infty)$ to define the space $\mathcal{W}(I)$, but the trace space $V = Tr(\mathcal{W})$ will always be the same.

It is easy to see that the norm $\|\cdot\|_{1/2}$ is really a norm and that with respect to this norm the space V is a Banach space.

Definition 3.4.4. Suppose that H, W are Hilbert spaces with dense and continuous inclusion $W \hookrightarrow H$. Let $A : W \rightarrow H$, be a linear operator that satisfies the following

i) A is self-adjoint with the domain $D(A) = W$, i.e.

$$\langle Ax, y \rangle_H = \langle x, Ay \rangle_H,$$

for all $x, y \in W = D(A)$. It follows from Hellinger-Toplitz theorem that A is also continuous.

ii) A is positive, $\langle Ax, x \rangle \geq 0$ for all $x \in W$.

iii) Suppose also that A is bijective. Hence, there exists a positive constant c_0 such that

$$\frac{1}{c_0} \|x\|_W \leq \|Ax\|_H \leq c_0 \|x\|_W \quad (3.61)$$

The right inequality follows from continuity of A , and left from open mapping theorem.

For $\theta \in [0, 1]$ we define the **intermediate (interpolation) space**

$$[W, H]_{1-\theta, A} = Dom(A^\theta).$$

Definition 3.4.5. We say that a self-adjoint operator $A : W \rightarrow H$ is a **purely point operator** (or has a **purely point spectrum**) if the following holds:

There exists an H -orthogonal decomposition $H = \bigoplus H_i$, where each $H_i = \langle e_i \rangle$, and e_i is an eigenvector of the operator A , i.e. $A(e_i) = \lambda_i e_i$.

A sufficient condition that a symmetric operator has eigenvectors which form a Hilbert space basis is that it has a compact inverse.

Remark 3.4.6. If A is a purely point operator which satisfies the requirements of the Definition 3.4.4, then for $H \ni \xi = \sum_i a_i e_i$ we have

$$[W, H]_{1-\theta, A} = \text{Dom}(A^\theta) = \left\{ \xi \in H, \xi = \sum_i a_i e_i : \sum_i \lambda_i^{2\theta} |a_i|^2 < +\infty \right\}.$$

One could do the same for an operator A which is not necessarily positive (Thus the condition *ii*) in definition 3.4.4 is superfluous. Namely, the operator $|A|$ is positive and self-adjoint and we can define

$$[W, H]_{1-\theta, A} = \text{Dom}(|A|^\theta) = \left\{ \xi \in H, \xi = \sum_i a_i e_i : \sum_i |\lambda_i|^{2\theta} |a_i|^2 < +\infty \right\}.$$

Notice that the space $[W, H]_{1-\theta, A}$ is a Hilbert space with the scalar product

$$\xi = \sum_i a_i e_i, \quad \eta = \sum_i b_i e_i, \quad \langle \xi, \eta \rangle_{1-\theta, A} = \sum_i a_i b_i |\lambda_i|^{2\theta}$$

Remark 3.4.7. In our intended applications of this theory the operator A will be the square root of the Laplacian, $A = \sqrt{-\Delta} = i\partial_t$ or some compact perturbation of $\sqrt{-\Delta}$. The spaces W and H will be some $H^k([0, 1]) = W^{k,2}([0, 1])$ spaces with certain boundary conditions.

In the next theorem we prove that the space $[W, H]_{1/2, A}$ doesn't depend on the operator A and that it is the same as the trace space $V = \text{Tr}(\mathcal{W})$.

Theorem 3.4.8. *Let A be an operator as in Remark 3.4.6 and let $\xi \in H$. Then $\xi \in [W, H]_{1/2, A} = D(\sqrt{A})$ if and only if $\xi \in V = \text{Tr}(\mathcal{W})$ and there exists a constant $c > 0$ such that for all $\xi \in D(\sqrt{A})$*

$$\frac{1}{c} \|\xi\|_{1/2} \leq \|\xi\|_{1/2, A} \leq c \|\xi\|_{1/2}.$$

Proof: We shall divide the proof of this theorem into three steps

Step 1: For all $x \in C_c^\infty([0, +\infty), W)$ we have that

$$\|x(0)\|_{1/2, A}^2 = \|\xi\|_{1/2, A}^2 \leq c \int_0^{+\infty} \left(\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2 \right) ds \quad (3.62)$$

Proof. As $W \subset [W, H]_{1/2, A}$ we have that $x(s) \in [W, H]_{1/2, A}$, for all $s \in [0, +\infty)$. Let $f = \dot{x}(s) + A(x(s))$, then

$$\begin{aligned} \frac{d}{ds} \frac{1}{2} \|x\|_{1/2, A}^2 &= \langle x, \dot{x} \rangle_{1/2, A} = \\ &= \langle x, f - Ax \rangle_{1/2, A} = \langle x, f \rangle_{1/2, A} - \|Ax\|_H^2 \end{aligned} \quad (3.63)$$

The last equality holds as $\langle Ax, x \rangle_{1/2,A} = \sum_{i=1}^{+\infty} \lambda_i^2 \langle x, e_i \rangle^2 = \|Ax\|_H^2$. Integrating the inequality (3.63) we obtain

$$\begin{aligned} \|\xi\|_{1/2,A}^2 &= \|x(0)\|_{1/2,A}^2 = \int_0^{+\infty} \|Ax\|_H^2 - \int_0^{+\infty} \langle x, f \rangle_A ds \\ &\leq \int_0^{+\infty} \|A(x(s))\|_H^2 dt + \frac{1}{2} \int_0^{+\infty} \|A(x(s))\|_H^2 ds + \frac{1}{2} \int_0^{+\infty} \|f\|_H^2 ds \\ &\leq \int_0^{+\infty} \left(\frac{1}{2} \|\dot{x}(s) + A(x(s))\|_H^2 + \frac{3}{2} \|A(x(s))\|_H^2 \right) ds \\ &\leq c_1^2 \int_0^1 \left(\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2 \right) ds \end{aligned}$$

and the constant $c_1 = \max \left\{ 1, \sqrt{\frac{5}{2}} c_0 \right\}$, where c_0 is the constant from the inequality (3.61). \square

Step 2: If $x \in L^2([0, +\infty), W) \cap W^{1,2}([0, +\infty), H) = \mathcal{W}$, then $\forall \tau \in [0, +\infty)$, $x(\tau) \in D(\sqrt{A})$.

Proof. For functions $x \in C_c^\infty([0, +\infty), W)$ this obviously holds, as $W \subset D(\sqrt{A})$. As the set $C_c^\infty([0, +\infty), W)$ is dense in \mathcal{W} choose sequence $x_k \rightarrow x$. The sequence x_k is Cauchy w.r.t. the norm $\int_0^{+\infty} \left(\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2 \right) dt$. Therefore this sequence is also Cauchy w.r.t. the norm $\|\cdot\|_{1/2,A}$ in $D(\sqrt{A})$ (Remark that the inequality proved in the first step holds for every $\tau \in [0, +\infty)$, not just $\tau = 0$). Therefore

$$x(\tau) = \lim_{k \rightarrow \infty} x_k(\tau) \in D(\sqrt{A}).$$

\square

Step 3: There exists $\delta > 0$ such that for all $\xi \in D(\sqrt{A})$ there exists $x \in \mathcal{V}$ with $x(0) = \xi$ and such that

$$\|\xi\|_{1/2,A}^2 \geq \delta \int_0^{+\infty} \left(\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2 \right) ds \quad (3.64)$$

Proof. Let $\xi = \sum_{i=1}^{+\infty} \xi_i$, $x(s) = \sum_{i=1}^{+\infty} x_i(s)$. As $\xi \in D(\sqrt{A})$ we have $\sum_i \lambda_i \xi_i^2 < +\infty$. The solution of the following problem

$$\dot{x}(s) + A(x(s)) = 0, \quad x(0) = \xi \quad (3.65)$$

is given by $x = \sum_i x_i(s) = \sum_i \xi_i e^{-\lambda_i s}$. We shall prove that $x \in \mathcal{W}$ and that

$$\|\xi\|_{1/2,A}^2 \geq c \int_0^{+\infty} \left(\|\dot{x}(s)\|_H^2 + \|A(x(s))\|_H^2 \right) ds, \quad (3.66)$$

for some positive constant c . As x satisfies the equality (3.65) it is enough to prove that $\int_0^{+\infty} \|A(x(s))\|_H ds < +\infty$. Notice that

$$\int_0^{+\infty} \|A(x_i)\|_H^2 ds = \int_0^{+\infty} \lambda_i^2 \xi_i^2 e^{-2\lambda_i s} ds \quad (3.67)$$

$$= \lambda_i \xi_i^2 \int_0^{+\infty} \lambda_i e^{-2\lambda_i s} ds = \frac{1}{2} \lambda_i \xi_i^2 \quad (3.68)$$

hence,

$$\int_0^{+\infty} \|A(x)\|_H^2 ds = \int_0^{+\infty} \sum_i \|A(x_i)\|_H^2 ds = \frac{1}{2} \|\xi\|_{1/2,A}^2.$$

As $\|A(x)\|_H \geq \frac{1}{c_0} \|x\|_W$ we have

$$\|\xi\|_{1/2,A}^2 = 2 \int_0^{+\infty} \|A(x)\|_H^2 ds \geq \delta \int_0^{+\infty} \left(\|\dot{x}(s)\|_H^2 + \|x(s)\|_W^2 \right) ds, \quad (3.69)$$

where δ behaves as $\frac{1}{c_0^2}$.

□

3.4.2 L^p estimates

In section 3.1 we have considered the linearized operator D_A given by

$$\begin{aligned} D_A &: W_{bc}^{2,2}(I \times [0, 1]) \rightarrow W_{bc}^{1,2}(I \times [0, 1]) \\ D_A \xi &= \partial_s \xi + A\xi = \partial_s \xi + J_0 \partial_t \xi + S(t)\xi, \end{aligned}$$

where the operator A satisfies (HA). Now we consider the same operator, but as the domain we take $W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$ defined by

$$W_{bc}^{1,p}(\mathbb{R} \times [0, 1]) := \left\{ \xi \in W^{1,p}(\mathbb{R} \times [0, 1]) \mid \xi(s, i) \in \mathbb{R}^n \times \{0\} \right\}. \quad (3.70)$$

The next lemma is an analogous of the Lemma 2.4 in [24] and the proof is almost verbatim taken from there.

Lemma 3.4.9. *Suppose that $S : [0, 1] \rightarrow R^{2n \times 2n}$ is a smooth function and suppose that the operator $A = J_0 \partial_t + S(t)$ satisfies (HA). Let $1 < p < \infty$ and let D_A be given by*

$$\begin{aligned} D_A &: W_{bc}^{1,p}(\mathbb{R} \times [0, 1]) \rightarrow L^p(\mathbb{R} \times [0, 1]) \\ D_A \xi &= \partial_s \xi + A \xi = \partial_s \xi + J_0 \partial_t \xi + S(t) \xi, \end{aligned}$$

is bijective and there exists a constant $c > 0$ such that the following inequality

$$\|\xi\|_{W^{1,p}} \leq c \|D_A \xi\|_{L^p}, \quad (3.71)$$

holds for all $\xi \in W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$.

Proof. Step 1. The claim holds in the case $p = 2$. Notice that

$$\begin{aligned} \|D_A \xi\|_{L^2}^2 &= \int_{-\infty}^{+\infty} \langle \partial_s \xi + A \xi, \partial_s \xi + A \xi \rangle_{L^2} ds \\ &= \int_{-\infty}^{+\infty} \left(\|\partial_s \xi\|_{L^2}^2 + \|A \xi\|_{L^2}^2 \right) + \int_{-\infty}^{+\infty} \partial_s \langle \xi, A \xi \rangle ds \\ &= \int_{-\infty}^{+\infty} \left(\|\partial_s \xi\|_{L^2}^2 + \|A \xi\|_{L^2}^2 \right) \\ &\geq c' \|\xi\|_{W^{1,2}(\mathbb{R} \times [0, 1])}. \end{aligned}$$

Thus, we have proved that the operator D_A is injective and has a closed image. To prove that it is also surjective we can use eigenvector decomposition. Let $\eta \in L^2(\mathbb{R} \times [0, 1])$. Write $\eta(s, t) = \sum_{\lambda} \eta_{\lambda}(s, t)$, where η_{λ} are the eigenvectors of the operator A . Observe the equation $D_A \xi = \partial_s \xi + A \xi = \eta$. Then the solution $\xi = \sum_{\lambda} \xi_{\lambda}$ of this equation is given by

$$\begin{aligned} \xi_{\lambda}(s, t) &= \int_{-\infty}^s e^{-\lambda(s-\tau)} \eta_{\lambda}(\tau, t) d\tau, \quad \lambda > 0 \\ \xi_{\lambda}(s, t) &= - \int_s^{\infty} e^{-\lambda(s-\tau)} \eta_{\lambda}(\tau, t) d\tau, \quad \lambda < 0 \end{aligned}$$

Notice that $\xi = \int_{-\infty}^{+\infty} K(s - \tau) \eta(\tau, t) d\tau$, where K decays exponentially.

Step 2. Let $p \geq 2$. There exists a constant $c_1 > 0$ such that

$$\|\xi\|_{W^{1,p}([0,1] \times [0,1])} \leq c_1 \left(\|D_A \xi\|_{L^p([-1,2] \times [0,1])} + \|\xi\|_{L^2([-1,2] \times [0,1])} \right) \quad (3.72)$$

holds for all $\xi \in W^{1,p}([-1, 2] \times [0, 1])$. Moreover, if $\xi \in W^{1,2}$ and $D\xi \in L^p_{\text{loc}}$ then $\xi \in W^{1,p}_{\text{loc}}$.

From Calderon-Zygmung inequality we have that

$$\|\nabla \xi\|_{L^p([0,1] \times [0,1])} \leq c \|\bar{\partial} \xi\|_{L^p([-1/2, 3/2] \times [0,1])}.$$

This implies the above estimate with L^2 norm on the right side replaced by L^p norm. Let $\Omega' = [-1/2, 3/2] \times [0, 1]$ and $\Omega = [-1, 2] \times [0, 1]$ we have

$$\begin{aligned} \|\xi\|_{W^{1,p}([0,1] \times [0,1])} &\leq c \left(\|D_A \xi\|_{L^p(\Omega')} + \|\xi\|_{L^p(\Omega')} \right) \\ &\leq c \left(\|D_A \xi\|_{L^p(\Omega')} + \|\xi\|_{W^{1,2}(\Omega')} \right) \\ &\leq c \left(\|D_A \xi\|_{L^p(\Omega')} + \|D_A \xi\|_{L^2(\Omega)} + \|\xi\|_{L^2(\Omega)} \right) \\ &\leq c \left(\|D_A \xi\|_{L^p(\Omega)} + \|\xi\|_{L^2(\Omega)} \right) \end{aligned}$$

The second inequality follows from Sobolev embedding.

Step 3. Consider the norm

$$\|\xi\|_{2,p} := \left(\int_{-\infty}^{\infty} \|\xi(s, \cdot)\|_{L^2([0,1])}^p ds \right)^{1/p}.$$

There exists constants $c_2, c_3 > 0$ such that, if $\xi \in W^{1,2}_{bc}(\mathbb{R} \times [0, 1])$ and $D\xi \in L^p(\mathbb{R} \times [0, 1])$, then $\xi \in W^{1,p}_{bc}(\mathbb{R} \times [0, 1])$ and

$$\|\xi\|_{2,p} \leq c_2 \|D\xi\|_{L^p}, \quad \|\xi\|_{W^{1,p}} \leq c_3 (\|D\xi\|_{L^p} + \|\xi\|_{2,p}).$$

From Step 1 it follows that $\xi \in W^{1,p}_{\text{loc}}$, to prove that $\xi \in W^{1,p}$ it is enough to prove the two estimates. The first inequality follows from Young's convolution inequality. From Step 1 we have that $\xi = K * \eta$, where $K(s)$ decays exponentially

$$\|\xi\|_{2,p} = \|K * \eta\|_{2,p} \leq \|K\|_{L^1(\mathbb{R}, L^2(0,1))} \|\eta\|_{L^p(\mathbb{R}, L^2(0,1))} \leq c_2 \|\eta\|_{L^p(\mathbb{R} \times [0,1])}$$

and the last inequality follows as $\|\eta\|_{L^2(0,1)} \leq \|\eta\|_{L^p(0,1)}$ for $p \geq 2$. To prove the second inequality, we shall use the result from Step 1 and the inequality

$$(a + b)^p \leq 2^p(a^p + b^p).$$

$$\begin{aligned} \|\xi\|_{W^{1,p}([k,k+1] \times [0,1])}^p &\leq 2^p c_1^p \left(\int_{k-1}^{k+2} \|D\xi\|_{L^p([0,1])}^p ds + \left(\int_{k-1}^{k+2} \|\xi\|_{L^2([0,1])}^2 ds \right)^{p/2} \right) \\ &\leq 2^p c_1^p \left(\int_{k-1}^{k+2} \|D\xi\|_{L^p([0,1])}^p ds + 3^{p/2-1} \int_{k-1}^{k+2} \|\xi\|_{L^2([0,1])}^p ds \right) \\ &\leq 3^{p/2} 2^p c_1^p \int_{k-1}^{k+2} (\|D\xi\|_{L^p}^p + \|\xi\|_{L^2}^p) ds \end{aligned}$$

Taking the sum over all $k \in \mathbb{Z}$ we obtain the desired inequality.

Step 4. Proof of Lemma for $p > 2$.

From Step 3, we have that

$$\|\xi\|_{1,p} \leq c \|D\xi\|_{L^p},$$

holds for all $\xi \in C_{bc}^\infty(\mathbb{R} \times [0, 1])$ with compact support. Thus by density of such functions the above inequality holds for all $\xi \in W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$. Thus the operator $D_A : W_{bc}^{1,p} \rightarrow L^p$ is injective and has closed image. Let $\eta \in L^p(\mathbb{R} \times [0, 1]) \cap L^2(\mathbb{R} \times [0, 1])$. By Step 1 there exists $\xi \in W_{bc}^{1,2}(\mathbb{R} \times [0, 1])$ such that $D\xi = \eta$. By Step 3 we have that $\xi \in W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$ and thus the image is dense and hence is onto.

Step 5. Case $1 < q < 2$.

By duality we have

$$\begin{aligned} \|D_A \xi\|_{L^q} &= \sup_{\|\eta\|_{L^p} \leq 1} \left| \int_{\Omega} \langle D_A \xi, \eta \rangle ds dt \right| \\ &\geq \sup_{\|\eta\|_{W^{1,p}} \leq 1} \left| \int_{\Omega} \langle D_A \xi, \eta \rangle ds dt \right| \\ &= \sup_{\|\eta\|_{W^{1,p}} \leq 1} \left| \int_{\Omega} \langle \xi, D_A^* \eta \rangle ds dt \right| \\ &= c \|\xi\|_{L^q} \end{aligned}$$

The last equality follows from the bijectivity of the operator $D_A^* : W_{bc}^{1,p} \rightarrow L^p$, where $D_A^* = -\partial_s + J_0 \partial_t + S(t)$. Particularly there exists η such that $\|\eta\|_{W^{1,p}} \leq 1$ and $D_A^* \eta = \frac{c|\xi|^{q-2}\bar{\xi}}{\|\xi\|_{L^q}^{q-1}}$. From the above inequality it follows that D_A is injective and analogously as in Step 4 we have that its image is dense. \square

Chapter 4

Hardy space approach to gluing

Symplectic Floer homology was introduced by Floer in [4, 5, 6, 7]. Floer Gluing theorem is one of the main technical ingredients in the construction of the Floer homology. Together with the compactness and linear elliptic Fredholm theory is used to define Floer homology and prove its various properties.

In this chapter we introduce a new approach to gluing of perturbed holomorphic curves with Lagrangian boundary conditions. The motivation comes on one hand from the work of Salamon, Robbin and Ruan in [22, 23], where similar techniques were used in integrable case, on the other hand from the work of Kronheimer and Mrowka [14] in the Seiberg-Witten setting.

Perturbed holomorphic curves with Lagrangian boundary conditions are solutions $u : I \times [0, 1] \rightarrow M$ of the Floer equation

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \quad u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \quad (4.1)$$

where L_i , $i = 0, 1$ are Lagrangian submanifolds of a symplectic manifold M , J_t is a smooth family of almost complex structures and $I \subset \mathbb{R}$ is an interval. The perturbation here comes in the form of a Hamiltonian vector field X_{H_t} . We shall be interested in the cases when the interval I is either a half infinite interval or $I = [-T, T]$. We introduce the moduli space \mathcal{M}^∞ consisting of the pairs of half-infinite perturbed holomorphic curves, i.e. solutions of (4.1), and analogously we introduce the moduli space \mathcal{M}^T of finite strips $u : [-T, T] \times [0, 1] \rightarrow M$ satisfying (4.1). The moduli space \mathcal{M}^T is a Hilbert submanifolds of certain Hilbert manifold of strips \mathcal{B}^T consisting of $W^{2,2}$ maps $u : I \times [0, 1] \rightarrow M$ satisfying the standard Lagrangian boundary condition

$$u(s, i) \in L_i, \quad i = 0, 1$$

and the condition on the first derivative

$$J_i(u(s, i))(\partial_t u(s, i) - X_{H_i}(u(s, i))) \in T_{u(s, i)} L_i, \quad i = 0, 1$$

and analogous result holds for $\mathcal{M}^\infty \subset \mathcal{B}^\infty$. This setup is needed in order to establish the necessary estimates for the nonlinear Hardy spaces.

In section 4.2, we introduce the Hilbert manifolds of strips and the Hilbert manifold $\mathcal{P}^{3/2}$ consisting of $W^{3/2,2}$ paths that satisfy Lagrangian boundary conditions and a certain condition on the first derivative. The path space $\mathcal{P}^{3/2}$ is actually the trace manifold of the Hilbert manifold of strips \mathcal{B}^T , namely restricting an element $u \in \mathcal{B}^T$ to the free boundary $u(\pm T, \cdot)$ we obtain an element of $\mathcal{P}^{3/2}$. For a Hamiltonian path $x \in \mathcal{P}^{3/2}$, we denote by \mathcal{U} some small neighborhood of x within $\mathcal{P}^{3/2}$. We focus our attention to some subsets of the moduli spaces \mathcal{M}^∞ and \mathcal{M}^T , which we shall denote by $\mathcal{M}^\infty(\mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$. These subsets consist only of those elements of the aforementioned moduli spaces which have sufficiently small energy and are on the free boundary (for example $\{\pm T\} \times [0, 1]$) close to the given Hamiltonian path x . Thus we assume that $u(\pm T, \cdot) \in \mathcal{U}$, where \mathcal{U} is the aforementioned neighborhood of a Hamiltonian path.

In Theorem 2.1.4 we have proved monotonicity results for the solutions of (4.1). These results guarantee that small energy solutions of (4.1), that at the ends are close to the Hamiltonian path x are confined to a small neighborhood of x . This will imply that the elements of $\mathcal{M}^\infty(\mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$ are contained in a small neighborhood of x and will allow us to work in suitable coordinate charts, thus the main analysis can be done in the standard model in Euclidean space using suitable coordinate charts. Consider the restriction maps

$$\begin{aligned} i^T : \mathcal{M}^T &\rightarrow \mathcal{P}^{3/2} \times \mathcal{P}^{3/2}, \quad i^T(u) = (u(-T, \cdot), u(T, \cdot)) \\ i^\infty : \mathcal{M}^\infty &\rightarrow \mathcal{P}^{3/2} \times \mathcal{P}^{3/2}, \quad i^\infty(u^-, u^+) = (u^-(0, \cdot), u^+(0, \cdot)). \end{aligned}$$

We prove that these maps are injective immersions and the restrictions of i^∞ and i^T to $\mathcal{M}^\infty(\mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$ are embeddings. The images $\mathcal{W}^\infty = i^\infty(\mathcal{M}^\infty(\mathcal{U}))$ and $\mathcal{W}^T = i^T(\mathcal{M}^T(\mathcal{U}))$ are the **nonlinear Hardy spaces** of the title. In section 4.5 we prove that \mathcal{W}^T converge to \mathcal{W}^∞ in the C^1 topology. This is the main result of this chapter.

This chapter is organized as follows:

- i) In section 4.1 we explain the setup and state the main theorems.
- ii) In section 4.2 we prove that the sets of paths and strips, $\mathcal{P}^{3/2}$, \mathcal{B}^T and \mathcal{B}^∞ are indeed Hilbert manifolds and we explicitly construct coordinate charts on these manifolds.

- iii) Sections 4.3, 4.4 and 4.5 contain the proofs of the main theorems. They also rely on linear elliptic estimates from the previous chapter. The hard of the proof is the convergence theorem, proved in section 4.5. In the appendix 4.6 we recall some properties of Lions- Magenes interpolation which we use in various places within the thesis.

4.1 Main results

In this section we explain the setup and state the main theorems. Let (M, ω) be a symplectic manifold without boundary and let

$$\mathcal{L} = \mathcal{L}(M, \omega)$$

be the set of all compact Lagrangian submanifolds $L \subset M$ without boundary. Throughout we abbreviate

$$\mathbb{R}^+ := [0, \infty), \quad \mathbb{R}^- := (-\infty, 0].$$

4.1.1 (Hamiltonian Paths). Let $L_0, L_1 \in \mathcal{L}(M, \omega)$. Denote by $\mathcal{H}(M)$ the space of smooth functions $H : [0, 1] \times M \rightarrow \mathbb{R}$ and by $\mathcal{J}(M, \omega)$ the space of smooth families of ω -compatible almost complex structures $J = \{J_t\}_{0 \leq t \leq 1}$ on M . For $H \in \mathcal{H}(M)$ denote $H_t := H(t, \cdot)$ for $0 \leq t \leq 1$ and let $[0, 1] \rightarrow \text{Diff}(M, \omega) : t \mapsto \phi_t$ be the Hamiltonian isotopy generated by H via

$$\partial_t \phi_t = X_{H_t} \circ \phi_t, \quad \phi_0 = \text{id}, \quad \iota(X_{H_t})\omega = dH_t. \quad (4.2)$$

A Hamiltonian function H is called **regular** for (L_0, L_1) if the Lagrangian submanifolds L_0 and $\phi_1^{-1}(L_1)$ intersect transversally. The set of regular Hamiltonian functions will be denoted by $\mathcal{H}_{\text{reg}}(M, L_0, L_1)$. Intersection points of L_0 and $\phi_1^{-1}(L_1)$ correspond to solutions $x : [0, 1] \rightarrow M$ of Hamilton's equation

$$\dot{x}(t) = X_{H_t}(x(t)), \quad x(0) \in L_0, \quad x(1) \in L_1. \quad (4.3)$$

Denote the set of solutions of (4.3) by

$$\mathcal{C}(L_0, L_1; H) := \{x : [0, 1] \rightarrow M \mid x \text{ satisfies (4.3)}\}. \quad (4.4)$$

The set $\mathcal{C}(L_0, L_1; H)$ can be also seen as the set of critical points of the perturbed symplectic action functional on the space \mathcal{P} of paths in M connecting L_0 to L_1 .

4.1.2 (Floer Equation). Fix a regular Hamiltonian function $H = \{H_t\}_{0 \leq t \leq 1}$ and a smooth family of almost complex structures $J = \{J_t\}_{0 \leq t \leq 1} \in \mathcal{J}(M, \omega)$ and $L_0, L_1 \in \mathcal{L}(M, \omega)$. For a smooth map $u : \mathbb{R} \times [0, 1] \rightarrow M$ the **Floer equation** has the form

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \quad u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \quad (4.5)$$

The **energy** of a solution u of (4.5) is defined by

$$E_H(u) := \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(|\partial_s u|_t^2 + |\partial_t u - X_{H_t}(u)|_t^2 \right) dt ds.$$

Here $\langle \xi, \eta \rangle_t := \omega(\xi, J_t \eta)$ denotes the Riemannian metric determined by ω and J_t . If the energy is finite then the limits

$$x^\pm(t) := \lim_{s \rightarrow \pm\infty} u(s, t) \quad (4.6)$$

exist and belong to $\mathcal{C}(L_0, L_1; H)$ (see for example [21] and Proposition 2.3.1). The convergence is with all derivatives, uniform in t , and exponential.

4.1.3 (Hilbert Manifold of Paths). Let $L_0, L_1 \in \mathcal{L}(M, \omega)$ and let \mathcal{P}_L^1 denote the Hilbert manifold of paths with Lagrangian boundary conditions. More precisely

$$\mathcal{P}_L^1 := \{ \gamma \in W^{1,2}([0, 1], M) \mid \gamma(i) \in L_i, \ i = 0, 1 \}. \quad (4.7)$$

We prove in section 4.2 that this set is a Hilbert manifold. Consider the following two Hilbert space bundles over \mathcal{P}_L^1 .

$$\begin{array}{ccc} \mathcal{E}^1 & \xrightarrow{\quad} & \mathcal{E}^0 \\ & \searrow \quad \swarrow & \\ & \mathcal{P}_L^1 & \end{array} \quad (4.8)$$

with fibers

$$\begin{aligned} \mathcal{E}_\gamma^0 &= L^2([0, 1], \gamma^* TM) \\ \mathcal{E}_\gamma^1 &= \{ \xi \in W^{1,2}([0, 1], \gamma^* TM) \mid \xi(i) \in T_{\gamma(i)} L_i, \ i = 0, 1 \}. \end{aligned}$$

Let $\mathcal{E}_\gamma^{1/2}$ be the following interpolation space

$$\mathcal{E}_\gamma^{1/2} = [\mathcal{E}_\gamma^1, \mathcal{E}_\gamma^0]_{1/2}$$

In the appendix 4.6 we explain more about the interpolation theory relevant for our setting. The almost complex structure $J \in \mathcal{J}(M, \omega)$ and Hamiltonian $H \in \mathcal{H}(M)$ determine a section $\mathcal{S} : \mathcal{P}_L^1 \rightarrow \mathcal{E}^0$ via

$$\mathcal{S}(\gamma)(t) = J_t(\gamma)(\dot{\gamma}(t) - X_{H_t}(\gamma(t)))$$

Denote by $\mathcal{P}^{3/2}(H, J)$ the following set

$$\mathcal{P}^{3/2}(H, J) = \{\gamma \in \mathcal{P}_L^1 \mid \mathcal{S}(\gamma) \in \mathcal{E}_\gamma^{1/2}\}. \quad (4.9)$$

We prove in 4.2.9 that the set $\mathcal{P}^{3/2}(H, J)$ is a Hilbert manifold. We also give some more details about the interpolation space on which $\mathcal{P}^{3/2}(H, J)$ is modelled.

4.1.4 (Hilbert Manifolds of Strips). Fix two Lagrangian submanifolds $L_0, L_1 \in \mathcal{L}(M, \omega)$, fix regular Hamiltonian function $H \in \mathcal{H}_{\text{reg}}(M, L_0, L_1)$ (see 4.1.1) and almost complex structure $J \in \mathcal{J}(M, \omega)$. Let $x \in \mathcal{C}(L_0, L_1; H)$ be a Hamiltonian path. Observe the moduli space of infinite holomorphic strips, i.e. stable and unstable manifold defined as follows

$$\mathcal{M}^\pm(x; H, J) = \left\{ u \in W_{loc}^{2,2}(\mathbb{R}^\pm \times [0, 1], M) \left| \begin{array}{l} u \text{ satisfies (4.5),} \\ E_H(u) < +\infty \\ \lim_{s \rightarrow \pm\infty} u(s, t) = x(t) \end{array} \right. \right\}, \quad (4.10)$$

and the moduli space of finite strips

$$\mathcal{M}^T(H, J) = \{u \in W^{2,2}([-T, T] \times [0, 1], M) \mid u \text{ satisfies (4.5)}\}. \quad (4.11)$$

We shall prove that the moduli spaces $\mathcal{M}^\pm(x; H, J)$ and $\mathcal{M}^T(H, J)$ are Hilbert manifolds. Their ambient manifolds are Hilbert manifolds of strips $\mathcal{B}^\pm(x)$ and \mathcal{B}^T with the following boundary conditions

$$\begin{aligned} u(s, i) &\in L_i, \quad i = 0, 1 \\ J_i(u(s, i))(\partial_t u(s, i) - X_{H_t}(u(s, i))) &\in T_{u(s, i)}L_i, \quad i = 0, 1 \end{aligned} \quad (4.12)$$

Thus

$$\mathcal{B}^\pm(x) = \left\{ u \in W_{loc}^{2,2}(\mathbb{R}^\pm \times [0, 1], M) \left| \begin{array}{l} u \text{ satisfies (4.12)} \\ \lim_{s \rightarrow \pm\infty} u(s, t) = x(t) \end{array} \right. \right\}$$

and analogously

$$\mathcal{B}^T = \{u \in W^{2,2}([-T, T] \times [0, 1], M) \mid u \text{ satisfies (4.12)}\}.$$

We prove in 4.2.11 that \mathcal{B}^T and \mathcal{B}^\pm are Hilbert manifolds and in Section 4.3 we prove the following theorem.

Theorem 4.1.5. *Let $\mathcal{M}^\pm(H, J)$ and $\mathcal{M}^T(H, J)$ be defined by (4.10) and (4.11) and let \mathcal{B}^\pm and \mathcal{B}^T be as above.*

- a) *The sets $\mathcal{M}^\pm(x; H, J)$ and $\mathcal{M}^T(H, J)$ are Hilbert submanifolds of the Hilbert manifolds $\mathcal{B}^\pm(x)$ and \mathcal{B}^T respectively.*
- b) *The maps i^\pm and i^T defined by*

$$\begin{aligned} i^\pm &: \mathcal{M}^\pm(x; H, J) \rightarrow \mathcal{P}^{3/2}(H, J) \\ i^\pm(u) &= u(0, \cdot) \\ i^T &: \mathcal{M}^T(H, J) \rightarrow \mathcal{P}^{3/2}(H, J) \times \mathcal{P}^{3/2}(H, J) \\ u &\mapsto (u(-T, \cdot), u(T, \cdot)) \end{aligned} \tag{4.13}$$

are injective immersions.

Proof. See section 4.3. □

Denote by $\mathcal{M}^\infty(x; H, J)$ the product of stable and unstable manifolds

$$\mathcal{M}^\infty(x; H, J) = \mathcal{M}^+(x; H, J) \times \mathcal{M}^-(x; H, J). \tag{4.14}$$

and denote by i^∞ the product of the maps i^\pm

$$\begin{aligned} i^\infty &: \mathcal{M}^\infty(x; H, J) \rightarrow \mathcal{P}^{3/2}(H, J) \times \mathcal{P}^{3/2}(H, J) \\ i^\infty(u^+, u^-) &= (i^+(u^+), i^-(u^-)) = (u^+(0, \cdot), u^-(0, \cdot)). \end{aligned} \tag{4.15}$$

It follows from Theorem 4.1.5 that the mapping i^∞ is an injective immersion, as a product of injective immersions.

4.1.6 (Convergence and embedding theorem). Let the moduli spaces $\mathcal{M}^T(H, J)$ and $\mathcal{M}^\infty(x; H, J)$ be defined by (4.11) and (4.14). We consider only those elements of these moduli spaces which have sufficiently small energy and which are sufficiently close on the boundary to a Hamiltonian path $x \in \mathcal{C}(L_0, L_1; H)$. To explain this more precisely we fix some small neighborhood U of a point $p = x(0) \in L_0 \cap \phi_1^{-1}(L_1)$ that doesn't contain any other intersection points of $L_0 \cap \phi_1^{-1}(L_1)$. We have assumed that the Hamiltonian $H \in \mathcal{H}_{\text{reg}}(L_0, L_1; M)$ thus the intersection $L_0 \cap \phi_1^{-1}(L_1)$ is transverse and compact. Let $U_t = \phi_t(U)$, where ϕ_t is Hamiltonian isotopy (4.2). Let $V \subset \overline{V} \subset U$ be a neighborhood of $x(0)$ and let $V_t = \phi_t(V)$. Suppose that on U_t we have constructed coordinate charts $f_t : U_t \rightarrow \mathbb{R}^{2n}$. We construct such family of coordinate charts in section 4.2.

By monotonicity result, Theorem 2.1.4, there exists a constant $\hbar > 0$ such that every solution $u : [-T, T] \times [0, 1] \rightarrow M$ of (4.5) with $E(u) < \hbar$ and

$u(\pm T, t) \in V_t$ satisfies $u(s, t) \in U_t$ for every s, t . The analogous result holds for half-infinite holomorphic strips $u \in \mathcal{M}^\pm(x; H, J)$ with $u(0, t) \in V_t$.

Let \mathcal{U} be a neighborhood of a Hamiltonian path $x \in \mathcal{P}^{3/2}(H, J)$ in the Hilbert manifold $\mathcal{P}^{3/2}(H, J)$ defined in (4.9). Shrinking \mathcal{U} if necessary we may assume that every $\gamma \in \mathcal{U}$ satisfies $\gamma(t) \in V_t$, where V_t is as above. For \hbar and \mathcal{U} as above we define the following sets

$$\begin{aligned}\mathcal{M}^\infty(x, \mathcal{U}) &= \{(u^+, u^-) \in \mathcal{M}^\infty(x; H, J) \mid u^\pm(0, \cdot) \in \mathcal{U}, E(u^\pm) < \hbar\}, \\ \mathcal{M}^T(\mathcal{U}) &= \{u \in \mathcal{M}^T(H, J) \mid u(\pm T, \cdot) \in \mathcal{U}, E(u) < \hbar\}.\end{aligned}\tag{4.16}$$

Here $\mathcal{M}^\infty(x; H, J)$ is defined by (4.14) and $\mathcal{M}^T(H, J)$ by (4.11). All holomorphic curves $u \in \mathcal{M}^T(\mathcal{U})$ are contained in coordinate charts $U_t = \phi_t(U)$ and the same holds for $(u^+, u^-) \in \mathcal{M}^\infty(x, \mathcal{U})$ too. Thus, instead of working with holomorphic curves in M we can work in local coordinates in \mathbb{R}^{2n} , which is much simpler for the analysis. The main theorems are the following:

Theorem 4.1.7. *Let $\mathcal{M}^\infty(x, \mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$ be defined by (4.16). Let i^∞ and i^T be defined by (4.13) and (4.15). There exists an open neighborhood \mathcal{U} of a Hamiltonian path x such that the restrictions of the maps i^∞ and i^T to $\mathcal{M}^\infty(x, \mathcal{U})$ and $\mathcal{M}^T(\mathcal{U})$ are embeddings for all $T \geq 1$.*

Proof. See section 4.4. □

Theorem 4.1.8. *Assume the notation as in Theorem 4.1.7. Let*

$$\mathcal{W}^\infty(x, \mathcal{U}) := i^\infty(\mathcal{M}^\infty(x, \mathcal{U})) \text{ and } \mathcal{W}^T(\mathcal{U}) := i^T(\mathcal{M}^T(\mathcal{U})).$$

Then after possibly shrinking the neighborhood \mathcal{U} the manifolds $\mathcal{W}^T(\mathcal{U})$ converge to $\mathcal{W}^\infty(x, \mathcal{U})$ in the C^1 -topology.

Proof. See section 4.5. □

Remark 4.1.9. It is enough to prove Theorems 4.1.7 and 4.1.8 in the case that the Hamiltonian is identically equal zero. Let ϕ_t be the Hamiltonian isotopy as in (4.2). By naturality, we can consider the tuple

$$\tilde{u}(s, t) = \phi_t^{-1}(u(s, t)), \quad \tilde{L}_0 = L_0, \quad \tilde{L}_1 = \phi_1^{-1}(L_1), \quad \tilde{J}_t = \phi_t^* J_t.\tag{4.17}$$

If u is the solution of the equation (4.5), then $\tilde{u} : \mathbb{R}^\pm \times [0, 1] \rightarrow M$ satisfies unperturbed **Floer** equation, i.e. \tilde{u} is a \tilde{J} holomorphic curve,

$$\partial_s \tilde{u} + \tilde{J}_t(\tilde{u}) \partial_t \tilde{u} = 0, \quad \tilde{u}(s, 0) \in \tilde{L}_0, \quad \tilde{u}(s, 1) \in \tilde{L}_1.\tag{4.18}$$

By naturality, to Hamiltonian paths $x \in \mathcal{C}(L_0, L_1; H)$ correspond intersection points $\tilde{x} = x(0) \in \tilde{L}_0 \cap \tilde{L}_1$. By assumption the Hamiltonian H is regular, thus the intersection $\tilde{L}_0 \cap \tilde{L}_1$ is transverse. Hence, every solution $\tilde{u} : \mathbb{R}^\pm \times [0, 1] \rightarrow M$ of the equation (??) with finite energy converges exponentially to an intersection point $\tilde{x} \in \tilde{L}_0 \cap \tilde{L}_1$ (see Proposition 2.3.1 for the case of tame almost complex structure and the clean intersection of Lagrangian submanifolds).

4.2 Hilbert manifold setup

4.2.1 (Hilbert manifold of paths with Lagrangian boundary conditions). In this section we introduce a collection of Hilbert manifolds of paths with Lagrangian boundary conditions. We prove that they are infinite dimensional Hilbert manifolds and we explicitly construct coordinate charts on them. Fix $L_0, L_1 \in \mathcal{L}(M)$ and a regular Hamiltonian $H \in \mathcal{H}_{\text{reg}}(M, L_0, L_1)$.

Proposition 4.2.2. (i) *The set*

$$\mathcal{P}_L^1 := \{\gamma \in W^{1,2}([0, 1], M) \mid \gamma(0) \in L_0, \gamma(1) \in L_1\}. \quad (4.19)$$

is a smooth Hilbert submanifold of $W^{1,2}([0, 1], M)$.

(ii) *The set*

$$\mathcal{P}_L^2 := \left\{ \gamma \in W^{2,2}([0, 1], M) \left| \begin{array}{l} \gamma(0) \in L_0, \gamma(1) \in L_1, \\ J_0(\gamma(0))(\dot{\gamma}(0) - X_{H_0}(\gamma(0))) \in T_{\gamma(0)}L_0 \\ J_1(\gamma(1))(\dot{\gamma}(1) - X_{H_1}(\gamma(1))) \in T_{\gamma(1)}L_1 \end{array} \right. \right\}. \quad (4.20)$$

is a smooth Hilbert submanifold of $W^{2,2}([0, 1], M)$.

Proof. Denote by $\mathcal{P}^k = W^{k,2}([0, 1], M)$. The set \mathcal{P}^k is a smooth Hilbert manifold modeled on $W^{k,2}(\gamma^*TM)$ and local charts are given via exponential map. Denote by

$$\text{ev}_M : \mathcal{P}^1 \rightarrow M \times M$$

the evaluation map defined by

$$\text{ev}_M(\gamma) := (\gamma(0), \gamma(1)).$$

This map is smooth and its first derivative at γ is the linear map

$$\text{dev}_M(\gamma) : W^{1,2}([0, 1], \gamma^*TM) \rightarrow T_{\gamma(0)}M \times T_{\gamma(1)}M,$$

given by

$$\text{dev}_M(\gamma)\xi = (\xi(0), \xi(1)).$$

This map is clearly surjective for every $\gamma \in \mathcal{P}^1$. Hence ev_M is a submersion and hence the preimage

$$\mathcal{P}_L^1 = \text{ev}_M^{-1}(L_0 \times L_1)$$

is a smooth submanifold of \mathcal{P}^1 whose tangent space at γ is the preimage of $T_{\gamma(0)}L_0 \times T_{\gamma(1)}L_1$ under $\text{dev}_M(\gamma)$. This proves (i).

To prove (ii) we define the evaluation map

$$\text{ev}_{TM} : \mathcal{P}^2 \rightarrow TM \times TM$$

by

$$\text{ev}_{TM}(\gamma) = (\gamma(0), v_0, \gamma(1), v_1)$$

where $v_i = J_i(\gamma(i))(\dot{\gamma}(i) - X_{H_i}(\gamma(i)))$, $i = 0, 1$. Then

$$\mathcal{P}_L^k = \text{ev}_{TM}^{-1}(TL_0 \times TL_1).$$

We prove that the mapping ev_{TM} is also a submersion and hence \mathcal{P}_L^2 is a Hilbert submanifold of \mathcal{P}^2 . Choose a Riemannian metric on M . Fix a point $p \in M$ and a tangent vector $v \in T_pM$. The Riemannian metric determines an isomorphism

$$T_{(p,v)}TM \cong T_pM \oplus T_pM$$

as follows. Think of a tangent vector of TM at (p, v) as an equivalence class of curves $\mathbb{R} \rightarrow TM : t \mapsto (\gamma(t), X(t))$ with $\gamma(0) = p$ and $X(0) = v$, where two curves are equivalent iff they have the same derivative at $t = 0$ in some (and hence every) coordinate chart on TM containing the point (p, v) . The isomorphism $T_{(p,v)}TM \rightarrow T_pM \oplus T_pM$ is then given by

$$[\gamma, X] \mapsto (\dot{\gamma}(0), \nabla X(0))$$

where ∇X denotes the covariant derivative of a vector field along γ . With this understood the derivative of the map ev_{TM} is given by

$$\text{dev}_{TM}(\gamma)\xi = ((\xi(0), \eta(0)), (\xi(1), \eta(1))),$$

Denote for $i = 0, 1$

$$\begin{aligned} \zeta_i &= \left(\nabla_{\xi(i)} J_i(\gamma(i)) \right) \left(\dot{\gamma}(i) - X_{H_i}(\gamma(i)) \right) + J_i(\gamma(i)) \nabla \xi(i) \\ \mu_i &= J_i(\gamma(i)) \left(\nabla_{\xi(i)} X_{H_i}(\gamma(i)) \right) \end{aligned}$$

Then we have

$$\eta(0) = \zeta_0 - \mu_0, \quad \eta(1) = \zeta_1 - \mu_1$$

This map is surjective because $J_0(\gamma(0))$ and $J_1(\gamma(1))$ are isomorphisms of the respective tangent space of M . This proves the proposition. \square

Another way to prove that \mathcal{P}_L^1 and \mathcal{P}_L^2 are Hilbert manifolds is to construct explicitly coordinate charts on them. In the case of \mathcal{P}_L^1 one can use only the exponential map of an appropriately chosen metric g_t which has the property that L_i are totally geodesic with respect to g_i for $i = 0, 1$ (see Lemma 2.3.5 for a construction of such metrics). In this case we obtain coordinate charts which map \mathcal{P}_L^1 into its model space, i.e.

$$W_{bc}^1([0, 1], \mathbb{R}^{2n}) = \{ \xi \in W^{1,2}([0, 1], \mathbb{R}^{2n}) \mid \xi(i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \}. \quad (4.21)$$

In the case of \mathcal{P}_L^2 one cannot hope to find a metric such that the local charts are given only by its exponential map. This is true because the definition of \mathcal{P}_L^2 involves two boundary conditions, one of these conditions includes the derivative of a curve γ and we don't have enough control of the derivative of the exponential map. Still, we can explicitly construct local charts that map \mathcal{P}_L^2 into its model space, i.e. into

$$W_{bc}^{2,2}([0, 1]) = \left\{ \xi \in W^{k,2}([0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \\ \partial_t \xi(i) \in \{0\} \times \mathbb{R}^n, i = 0, 1 \end{array} \right\}. \quad (4.22)$$

We shall first construct local coordinate charts in the case that Hamiltonian $H = 0$ and then we shall reduce the case $H \neq 0$ to the case $H = 0$ by naturality.

Lemma 4.2.3. *Let $L_0, L_1 \in \mathcal{L}(M)$, let $J \in \mathcal{J}(M, \omega)$ and let $\alpha : [0, 1] \rightarrow M$ be a smooth path with Lagrangian boundary conditions, $\alpha(i) \in L_i$, $i = 0, 1$. There is an open set $\tilde{U} \subset [0, 1] \times M$ and a smooth map $f : \tilde{U} \rightarrow \mathbb{R}^{2n}$ such that the following holds:*

i)

$$\alpha(t) \in \tilde{U}_t := \{ p \in M \mid (t, p) \in \tilde{U} \}.$$

ii) *The map $f_t(\cdot) := f(t, \cdot) : \tilde{U}_t \rightarrow f_t(\tilde{U}_t) = W_t$ is a diffeomorphism for every t and it satisfies:*

$$\begin{aligned} f_i(L_i \cap \tilde{U}_i) &= (\mathbb{R}^n \times \{0\}) \cap W_i, i = 0, 1 \\ ((f_i)_* J_i)(x, 0) &= J_{std}, (x, 0) \in W_i, i = 0, 1. \end{aligned} \quad (4.23)$$

iii) $\partial_t f_t(p) = 0$ for $t = 0, 1$ for all $p \in \tilde{U}_t$.

iv) *If $\alpha(t)$ is a constant path $\alpha(t) = p$ then \tilde{U} can be chosen such that \tilde{U}_t is independent of t .*

Proof of Lemma 4.2.3. We devide the proof into four steps.

Step 1. Construction of the trivialization of α^*TM :

There exists a smooth map

$$e_\alpha : [0, 1] \times \mathbb{R}^{2n} \rightarrow \alpha^*TM$$

such that

- i) $e_\alpha(t) : \mathbb{R}^{2n} \rightarrow T_{\alpha(t)}M$ is a vector space isomorphism for all t .
- ii) $e_\alpha(t)J_{\text{std}} = J_t(\alpha(t))e_{\alpha(t)}$ for all t , where J_{std} denotes standard complex structure in \mathbb{R}^{2n} .
- iii) $\omega(e_\alpha(t)\cdot, e_\alpha(t)\cdot) = \omega_{\text{std}}(\cdot, \cdot)$, where ω_{std} denotes the standard symplectic form in \mathbb{R}^{2n} .
- iv) $e_\alpha(i) : \mathbb{R}^n \times \{0\} \rightarrow T_{\alpha(i)}L_i$, $i = 0, 1$

Let $\{e_i\}_{1 \leq i \leq n}$ be the standard basis of $\mathbb{R}^n \times \{0\}$, then $\{e_i, J_{\text{std}}e_i\}_{1 \leq i \leq n}$ is the standard basis of \mathbb{R}^{2n} . We first construct a trivialization \tilde{e}_α of α^*TM which satisfies the conditions i) – iii). Let g_t be the metric obtained by pairing ω and J_t , $g_t(\cdot, \cdot) = \omega(\cdot, J_t(\cdot))$.

Let $\{v_i\}_{1 \leq i \leq n}$ be an orthonormal basis of $T_{\alpha(0)}L_0$ with respect to g_0 . We define

$$\tilde{e}_\alpha(0)e_i = v_i, \quad \tilde{e}_\alpha(0)(J_{\text{std}}e_i) = J_0(\alpha(0))v_i, \quad \text{for } i = 1, \dots, n$$

and extend $\tilde{e}_\alpha(0)$ linearly. We define $A_t(p) : T_pM \rightarrow T_pM$ as follows

$$g_t(p)(A_t(p)\cdot, \cdot) = \frac{1}{2}(\partial_t g_t)(p)(\cdot, \cdot). \quad (4.24)$$

We define

$$\tilde{e}_\alpha(t)e_i := v_i(t), \quad i = 1, \dots, n$$

where $v_i(t)$ is a solution of the following equation

$$\begin{aligned} \tilde{\nabla}_t v_i(t) &= -\frac{1}{2}A_t(\alpha)v_i(t) + \frac{1}{2}J_t(\alpha)A_t(\alpha)J_t(\alpha)v_i(t), \\ v_i(0) &= v_i, \end{aligned} \quad (4.25)$$

where $\tilde{\nabla}_t$ is given by

$$\tilde{\nabla}_t v := \nabla_t v - \frac{1}{2}J_t((\nabla \dot{\alpha} J_t)v + (\partial_t J_t)v),$$

$\nabla = \nabla^t$ is Levi-Civita connection of the metric g_t and A_t is defined by (4.24). Notice that both left and right side of the equation (4.25) commute with J_t . Thus if $v(t)$ is a solution of (4.25), then also $J_t(\alpha(t))v(t)$ is a solution of the same equation. Also if $v(t)$ and $w(t)$ are the solutions of the equation (4.25) then

$$0 = \frac{d}{dt} \left(g_t(v(t), w(t)) + g_t(J_t(\alpha)v(t), J_t(\alpha)w(t)) \right) = 2 \frac{d}{dt} g_t(v(t), w(t)).$$

Thus the vectors $\{v_i(t), J_t(\alpha(t))v_i(t)\}_{i=1,n}$ form an orthonormal (with respect to g_t) basis for all $t \in [0, 1]$. Thus we can define

$$\tilde{e}_\alpha(t)(J_{\text{std}}(e_i)) := J_t(\alpha(t))v_i(t), \quad i = 1, \dots, n$$

and extend it by linearity. Notice that the trivialization \tilde{e}_α satisfies the conditions *i) – iii)*, whereas the condition *iv)* doesn't have to be necessarily satisfied for $t = 1$. There exists some Lagrangian subspace $V \subset \mathbb{R}^{2n}$ such that

$$\tilde{e}_\alpha(1) : V \rightarrow T_{\alpha(1)}L_1.$$

There exists a smooth path $U(t)$ of unitary matrices with the property $U(0) = \text{Id}$ and $U(1) : \mathbb{R}^n \times \{0\} \rightarrow V$ (as $U(n)$ is connected). Thus the trivialization

$$e_\alpha(t) := \tilde{e}_\alpha(t)U(t),$$

satisfies all the required properties.

Step 2. Construction of local coordinate charts

Let h_t be a smooth family of metrics as in Lemma 2.3.5. Let $r_{\alpha(t),t}$ be the injectivity radius of the metric h_t at the point $\alpha(t)$. Let

$$W'_t = B_{r_{\alpha(t),t}}(0) \subset \mathbb{R}^{2n} \text{ and } U'_t = B_{r_{\alpha(t),t}}(\alpha(t)) \subset M. \quad (4.26)$$

Let $\psi'_t : W'_t \rightarrow U'_t$ be a smooth family of diffeomorphisms given by

$$\psi'_t(\xi) = \exp_{\alpha(t),t}(e_\alpha(t)\xi),$$

where $e_\alpha(t)$ is the trivialization constructed in Step 1.

The map

$$\phi'_t = (\psi'_t)^{-1} : U'_t \rightarrow W'_t \quad (4.27)$$

has the following properties:

$$\phi'_0(L_0 \cap U'_0) = W'_0 \cap (\mathbb{R}^n \times \{0\}), \quad \phi'_1(L_1 \cap U'_1) = W'_1 \cap (\mathbb{R}^n \times \{0\}).$$

Denote with J'_t the push forward of J_t via ϕ'_t , $J'_t := (\phi'_t)_* J_t$. Notice that as $d\phi'_t(\alpha(t)) = e_\alpha(t)^{-1}$, the almost complex structure

$$J'_t(0) = d\phi'_t(\alpha(t))J_t(\alpha(t))d\phi'_t(\alpha(t))^{-1}$$

satisfies

$$J'_t(0) = J_{std}, \quad \text{for all } t \in [0, 1]$$

In the next step we make J'_t standard on the whole $\mathbb{R}^n \times \{0\}$.

Step 3. Adapting the charts to the almost complex structure

There exist open sets $V, \tilde{V} \subset [0, 1] \times \mathbb{R}^{2n}$ such that

$$V_t = \{p | (t, p) \in V\}, \quad \tilde{V}_t = \{p | (t, p) \in \tilde{V}\}$$

are open neighborhoods of $0 \in \mathbb{R}^{2n}$. There exist a smooth map $\tilde{\Phi} : \tilde{V} \rightarrow V$ such that

$$\tilde{\Phi}_t = \tilde{\Phi}(t, \cdot) : \tilde{V}_t \rightarrow V_t,$$

is a diffeomorphism for every t and $\tilde{J}_t := (\tilde{\Phi}_t \circ \phi'_t)_* J_t$ satisfies

$$\tilde{J}_t(x, 0) = J_{std}, \quad \text{for all } (x, 0) \in V_t \cap (\mathbb{R}^n \times \{0\})$$

and ϕ'_t is a diffeomorphism from (4.27).

Let $\tilde{\Psi}_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be given by

$$\tilde{\Psi}_t(x, y) = \begin{pmatrix} x \\ 0 \end{pmatrix} + J'_t(x, 0) \begin{pmatrix} y \\ 0 \end{pmatrix},$$

where $J'_t = (\phi'_t)_* J_t$ and ϕ'_t is given by (4.27). Then

$$d\tilde{\Psi}_t(x, 0)(\hat{x}, \hat{y}) = \begin{pmatrix} \hat{x} \\ 0 \end{pmatrix} + J'_t(x, 0) \begin{pmatrix} \hat{y} \\ 0 \end{pmatrix}$$

and obviously $d\tilde{\Psi}_t(0, 0) = \mathbb{1}$ for all t . Hence there exist a smooth map $t \mapsto r_t > 0$ such that $\tilde{\Psi}_t : V_t = B_{r_t}(0) \rightarrow \tilde{V}_t = \tilde{\Psi}_t(V_t)$ is a diffeomorphism for every t . If necessary shrink \tilde{V}_t so that $\tilde{V}_t \subset W'_t$, where W'_t are as in (4.26). Notice that

$$d\tilde{\Psi}_t(x, 0) \circ J_{std} = J'_t(x, 0) d\tilde{\Psi}_t(x, 0)$$

for all $t \in [0, 1]$. The desired map $\tilde{\Phi}_t$ is the inverse of $\tilde{\Psi}_t$. The open set $\tilde{V} = \{(t, x) \mid x \in \tilde{V}_t\}$ and V is given analogously. This proves the second

step.

Step 4: Construction of the map f_t

Let $\tilde{U}_t = (\phi'_t)^{-1}(\tilde{V}_t)$, where \tilde{V}_t are the open sets as in Step 3 and ϕ'_t the map (4.27). The map $f_t : \tilde{U}_t \rightarrow V_t$ is given by

$$f_t = \tilde{\Phi}_t \circ \phi'_t.$$

Obviously, f_t satisfies (4.23). If $\partial_t f_t \neq 0$ for $t = 0, 1$ instead of f_t take $f_{\beta(t)}$, where $\beta(t)$ is a smooth cut-off function satisfying $\dot{\beta}(0) = \dot{\beta}(1) = 0$.

Finally in the case $\alpha(t) = p$ if it is necessary shrink V_t such that $f_t^{-1}(V_t) = U_p$, where U_p is some fixed neighborhood of the point p . \square

4.2.4 (Construction of Coordinate Charts on \mathcal{P}_L^i). Now we are able using the map f constructed in Lemma 4.2.3 to construct coordinate charts on \mathcal{P}_L^i , $i = 1, 2$. We shall first construct these charts in the case that the Hamiltonian function H is identically equal zero, and then in the general case, when H is an arbitrary function.

Corollary 4.2.5 (The case $H = 0$). *Assume that the Hamiltonian function $H = 0$ and let \mathcal{P}_L^i , $i = 1, 2$ be as in (4.19) and (4.20). Let $f_t : \tilde{U}_t \rightarrow \mathbb{R}^{2n}$ and α be as in Lemma 4.2.3. Define $\mathcal{U}_\alpha^i \subset \mathcal{P}_L^i$, $i = 1, 2$ and $\Phi_\alpha : \mathcal{U}_\alpha^i \rightarrow W_{bc}^{i,2}([0, 1], \mathbb{R}^{2n})$ by*

$$\mathcal{U}_\alpha^i = \left\{ \gamma \in \mathcal{P}_L^i \mid \gamma(t) \in \tilde{U}_t \right\}$$

and

$$\Phi_\alpha(\gamma)(t) = f_t(\gamma(t))$$

Then Φ_α is a coordinate chart on \mathcal{P}_L^i , $i = 1, 2$.

Corollary 4.2.6 (General Case). *Let \mathcal{P}_L^i , $i = 1, 2$ be as in (4.19) and (4.20). Let $f_t : \tilde{U}_t \rightarrow \mathbb{R}^{2n}$ and α be as in Lemma 4.2.3. Let ϕ_t be the Hamiltonian isotopy defined by (4.2) and denote $U_t = \phi_t(\tilde{U}_t)$. Define $\mathcal{U}_\alpha^i \subset \mathcal{P}_L^i$ and $\Phi_\alpha : \mathcal{U}_\alpha^i \rightarrow W_{bc}^{i,2}([0, 1], \mathbb{R}^{2n})$ by*

$$\mathcal{U}_\alpha^i = \{ \gamma \in \mathcal{P}_L^i \mid \gamma(t) \in U_t \}$$

and

$$\Phi_\alpha(\gamma)(t) = f_t(\phi_t^{-1}(\gamma(t))) = F_t(\gamma(t)).$$

Then $\Phi_\alpha : \mathcal{U}_\alpha^i \rightarrow W_{bc}^{i,2}([0, 1], \mathbb{R}^{2n})$ is a coordinate chart on \mathcal{P}_L^i , $i = 0, 1$.

4.2.7 (Hilbert space bundles \mathcal{E}^0 and \mathcal{E}^1). In the Proposition 4.2.2 we have introduced manifolds of paths \mathcal{P}_L^1 and \mathcal{P}_L^2 , but our main interest will be another Hilbert manifold, denoted by $\mathcal{P}^{3/2} = \mathcal{P}^{3/2}(H, J)$, which is in some sense an intermediate manifold between these two manifolds

$$\mathcal{P}_L^2 \subset \mathcal{P}^{3/2} \subset \mathcal{P}_L^1$$

Consider the following two Hilbert space bundles over \mathcal{P}_L^1

$$\begin{array}{ccc} \mathcal{E}^1 & \xrightarrow{\quad} & \mathcal{E}^0 \\ & \searrow \quad \swarrow & \\ & \mathcal{P}_L^1 & \end{array} \quad (4.28)$$

with fibers

$$\begin{aligned} \mathcal{E}_\gamma^0 &= L^2([0, 1], \gamma^*TM) \\ \mathcal{E}_\gamma^1 &= \{ \xi \in W^{1,2}([0, 1], \gamma^*TM) \mid \xi(i) \in T_{\gamma(i)}L_i, \ i = 0, 1 \}. \end{aligned} \quad (4.29)$$

Note that \mathcal{E}_γ^1 is a dense subset of \mathcal{E}_γ^0 and the inclusion of \mathcal{E}_γ^1 into \mathcal{E}_γ^0 is a compact operator. The tangent bundle of \mathcal{P}_L^1 is \mathcal{E}^1 . The almost complex structures $J \in \mathcal{J}(M, \omega)$ and Hamiltonian $H \in \mathcal{H}_{\text{reg}}(M, L_0, L_1)$ determine a section $\mathcal{S} : \mathcal{P}_L^1 \rightarrow \mathcal{E}^0$ via

$$\mathcal{S}(\gamma)(t) = J_t(\gamma) \left(\dot{\gamma}(t) - X_{H_t}(\gamma(t)) \right), \quad 0 \leq t \leq 1. \quad (4.30)$$

Notice that

$$\mathcal{P}_L^2 = \{ \gamma \in \mathcal{P}_L^1 \mid \mathcal{S}(\gamma) \in \mathcal{E}_\Lambda^1 \}.$$

4.2.8 (The interpolation subbundle $\mathcal{E}^{1/2}$). Let \mathcal{P}_L^i , $i = 1, 2$ be as in (4.19) and (4.20) and let \mathcal{E}_γ^i , $i = 0, 1$ be as in (4.29). Let

$$H := L^2([0, 1], \mathbb{R}^{2n}) \text{ and } W := W_{bc}^{1,2}([0, 1], \mathbb{R}^{2n})$$

be as in (4.21). Remember that the Hilbert manifold \mathcal{P}_L^1 is modelled on the Hilbert space W , whereas the Hilbert manifold \mathcal{P}_L^2 is modelled on the Hilbert space $W_{bc}^{2,2}([0, 1])$ defined by (4.22). There exist an isomorphism

$$e_\gamma : H \rightarrow \mathcal{E}_\gamma^0 \quad \text{and} \quad e_\gamma : W \rightarrow \mathcal{E}_\gamma^1.$$

There are many different ways to construct such isomorphism. Any local chart on \mathcal{P}_L^1 gives us a trivialization of its tangent bundle \mathcal{E}^1 , as well as of \mathcal{E}^0 . Thus we can use charts constructed in 4.2.4 or we can also use the

trivialization from the Step 1 of Lemma 4.2.3. Let $\mathcal{E}_\gamma^{1/2}$ be the following interpolation space

$$\mathcal{E}_\gamma^{1/2} = [\mathcal{E}_\gamma^1, \mathcal{E}_\gamma^0]_{1/2}. \quad (4.31)$$

It is characterized as the domain of the square root $A^{1/2}$ of any self-adjoint positive definite operator on \mathcal{E}_γ^0 with domain \mathcal{E}_γ^1 . Alternatively, it can be defined as the set of initial conditions $\xi(0) \in \mathcal{E}_\gamma^0$ of L^2 functions $\xi : [0, 1] \rightarrow \mathcal{E}_\gamma^1$ whose composition with inclusion $\mathcal{E}_\gamma^1 \hookrightarrow \mathcal{E}_\gamma^0$ is of class $W^{1,2}$. The Hilbert space $\mathcal{E}_\gamma^{1/2}$ is isomorphic via e_γ to the interpolation space

$$V := [W, H]_{1/2}.$$

More precisely, let $\xi = (\xi^1, \xi^2) \in V$, where ξ^1 denotes the first n coordinates and ξ^2 are the last n coordinates. Then evidently

$$\xi^1 \in [H^1, L^2]_{1/2} = H^{1/2}([0, 1], \mathbb{R}^n), \quad \xi^2 \in [H_0^1, L^2]_{1/2} = H_{00}^{1/2}([0, 1], \mathbb{R}^n), \quad (4.32)$$

where $H_{00}^{1/2}$ is Lions-Magenes' space [15]. We explain more in the Appendix 4.6 about the interpolation theory relevant for this setting. Thus, the Hilbert interpolation space $V = [W, H]_{1/2}$ is just the product $V = H^{1/2}([0, 1], \mathbb{R}^n) \times H_{00}^{1/2}([0, 1], \mathbb{R}^n)$, and

$$e_\gamma : V \xrightarrow{\cong} \mathcal{E}_\gamma^{1/2}.$$

Let $\mathcal{E}^{1/2}$ be the bundle over \mathcal{P}_L^1 with the fiber $\mathcal{E}_\gamma^{1/2}$ over γ .

$$\begin{array}{ccccc} \mathcal{E}^1 & \longrightarrow & \mathcal{E}^{1/2} & \longrightarrow & \mathcal{E}^0 \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{P}_L^1 & & \end{array} \quad (4.33)$$

4.2.9 (The Hilbert manifold $\mathcal{P}^{3/2}$). Let $\mathcal{E}_\gamma^{1/2}$ be as in (4.31) and let \mathcal{S} be defined (4.30). Denote by $\mathcal{P}^{3/2} = \mathcal{P}^{3/2}(H, J)$ the following set

$$\mathcal{P}^{3/2} = \mathcal{P}^{3/2}(H, J) := \{\gamma \in \mathcal{P}_L^1 \mid \mathcal{S}(\gamma) \in \mathcal{E}_\gamma^{1/2}\}. \quad (4.34)$$

We shall prove that the set $\mathcal{P}^{3/2}$ is a Hilbert manifold modelled on the following Hilbert space

$$H_{bc}^{3/2} := [W_{bc}^{2,2}([0, 1]), W_{bc}^{1,2}([0, 1])]_{1/2}, \quad (4.35)$$

where $W_{bc}^{i,2}([0, 1]) = W_{bc}^{i,2}([0, 1], \mathbb{R}^{2n})$, $i = 1, 2$ are as in (4.21) and (4.22). Notice that $H_{bc}^{3/2}$ can be written as the following space

$$H_{bc}^{3/2} = \left\{ (\xi_1, \xi_2) \in H^1([0, 1], \mathbb{R}^n) \times H_0^1([0, 1], \mathbb{R}^n) \mid \begin{array}{l} \partial_t \xi_1 \in H_{00}^{1/2}, \\ \partial_t \xi_2 \in H^{1/2} \end{array} \right\}. \quad (4.36)$$

Lemma 4.2.10. *Let $\mathcal{P}^{3/2}$ be defined by (4.34) and let $H_{bc}^{3/2}$ be as in (4.36). The set $\mathcal{P}^{3/2}$ is a Hilbert manifold modelled on the interpolation space $H_{bc}^{3/2}$. The Hilbert manifold structure is defined by the following construction. Let $\alpha \in \mathcal{P}_L^2 \subset \mathcal{P}^{3/2}$ be a smooth path and let $\{U_t\}_{0 \leq t \leq 1}$ be a smooth family of open sets as in Corollary 4.2.6. We define \mathcal{U}_α and $\Phi_\alpha : \mathcal{U}_\alpha \rightarrow H_{bc}^{3/2}$ as follows*

$$\mathcal{U}_\alpha = \{\gamma \in \mathcal{P}^{3/2} | \gamma(t) \in U_t, \forall t \in [0, 1]\}$$

and

$$\Phi_\alpha(\gamma)(t) = f_t(\phi_t^{-1}(\gamma(t))) = F_t(\gamma(t)), \quad (4.37)$$

where f_t is a family of diffeomorphisms constructed in Lemma 4.2.3 and ϕ_t is Hamiltonian isotopy (4.2). Then Φ_α is a local chart on \mathcal{U}_α .

Proof. Let $F_t = f_t \circ \phi_t^{-1}$ be as in equation (4.37) and let $\gamma \in \mathcal{U}_\alpha$. The mapping $F_t^* = dF_t(\gamma(t))^{-1}$ induces the following isomorphisms

$$F_t^* : H = L^2([0, 1], \mathbb{R}^{2n}) \rightarrow \mathcal{E}_\gamma^0, \quad F_t^* : W = W_{bc}^{1,2}([0, 1], \mathbb{R}^{2n}) \rightarrow \mathcal{E}_\gamma^1$$

Thus F_t^* induces the isomorphism

$$(F_t)^* : V = [W, H]_{1/2} \rightarrow \mathcal{E}_\gamma^{1/2}.$$

Let $\hat{\gamma}(t) = F_t(\gamma(t)) = (\xi_1(t), \xi_2(t))$, where ξ_1 denotes the first n -coordinates and ξ_2 the last n -coordinates. Then $\hat{\gamma}$ satisfies $\hat{\gamma}(i) \in \mathbb{R}^n \times \{0\}$, $i = 0, 1$. Thus we have

$$\xi_1 \in H^1([0, 1], \mathbb{R}^n), \quad \xi_2 \in H_0^1([0, 1], \mathbb{R}^n) \quad (4.38)$$

Denote by $\hat{J}_t = (F_t)_* J_t$. As $\mathcal{S}(\gamma) \in \mathcal{E}_\gamma^{1/2}$ we have $(F_t)_*(J_t(\gamma)(\partial_t \gamma - X_{H_t}(\gamma))) \in V$. The following equalities hold

$$\begin{aligned} (F_t)_* \left(J_t(\gamma)(\partial_t \gamma - X_{H_t}(\gamma)) \right) &= dF_t(\gamma(t)) J_t(\gamma(t)) \left(\partial_t \gamma(t) - X_{H_t}(\gamma) \right) = \\ ((F_t)_* J_t)(\hat{\gamma}) \left(dF_t(\gamma) \partial_t \gamma - dF_t(\gamma) X_{H_t} \right) &= \\ \hat{J}_t(\hat{\gamma}) \left(\partial_t \hat{\gamma} - (\partial_t F_t)(\gamma) - df_t \cdot d\phi_t^{-1} \cdot X_{H_t}(\gamma) \right) &= \\ \hat{J}_t(\hat{\gamma}) \left(\partial_t \hat{\gamma} - (\partial_t f_t) \phi_t^{-1}(\gamma) - df_t(\phi_t^{-1}(\gamma)) \partial_t \phi_t^{-1}(\gamma) - df_t(\phi_t^{-1}(\gamma)) d\phi_t^{-1}(\gamma) X_{H_t}(\gamma) \right) &= \\ \hat{J}_t(\hat{\gamma}) \left(\partial_t \hat{\gamma} - (\partial_t f_t)(\phi_t^{-1}(\gamma)) \right) &\in V \end{aligned} \quad (4.39)$$

From Lemma 4.2.3 we have that $\partial_t f_t = 0$ for $t = 0, 1$, this implies that

$$\hat{J}_t(\hat{\gamma})(\partial_t f_t)(\phi_t^{-1}(\gamma)) \in H_0^1([0, 1], \mathbb{R}^{2n}) \subset V$$

and from (4.39) it follows that

$$\hat{J}_t(\hat{\gamma})\partial_t\hat{\gamma} \in [W, H]_{1/2} = V = H^{1/2}([0, 1], \mathbb{R}^n) \times H_{00}^{1/2}([0, 1], \mathbb{R}^n) \quad (4.40)$$

From Lemma 4.2.3 we have that $\hat{J}_t(x, 0) = J_{\text{std}}$, $t = 0, 1$ for $x \in \mathbb{R}^n$, $|x| < r$, together with (4.40) and (4.32) this implies that

$$\partial_t \xi_1 \in [H_0^1, L^2]_{1/2} = H_{00}^{1/2}, \quad \partial_t \xi_2 \in [H^1, L^2]_{1/2} = H^{1/2}.$$

Thus we have $\hat{\gamma} \in H_{bc}^{3/2}$. Notice also that if we consider smooth paths α and β and local charts Φ_α and Φ_β given by (4.37), then the transition map $\Phi_{\beta\alpha} = \Phi_\beta \circ \Phi_\alpha^{-1}$ is given by

$$\Phi_{\beta\alpha}(\xi)(t) = f'_t \circ f_t^{-1}(\xi(t))$$

and f_t and f'_t are as in Lemma 4.2.3. As these maps are diffeomorphisms it follows that the transition maps are also diffeomorphisms. \square

4.2.11 (The Hilbert manifold of strips). Let $\mathcal{B}^\pm(x)$ and \mathcal{B}^T be defined as in 4.1.4. We prove that they are infinite dimensional Hilbert manifolds modelled on the following Hilbert spaces

$$W_{bc}^{2,2}(I \times [0, 1]) = \left\{ \xi \in W^{2,2}(I \times [0, 1], \mathbb{R}^{2n}) \left| \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \\ \partial_t \xi(s, i) \in \{0\} \times \mathbb{R}^n, \quad i = 0, 1 \end{array} \right. \right\} \quad (4.41)$$

where $I = \mathbb{R}^\pm$ in the case of infinite strips and $I = [-T, T]$ in the case of finite strips. We also prove that restricting an element $u \in \mathcal{B}^\pm(x)$ to the free (non Lagrangian) boundary we obtain an element of the Hilbert manifold of paths $\mathcal{P}^{3/2}$. The next result extends Lemma 4.2.3 to global strips.

Lemma 4.2.12. *Let $u \in \mathcal{B}^+(x)$ be a smooth strip such that $u(s, t) = x(t)$ for $s \geq s_0$, for some large s_0 . There exists an open set $U \subset \mathbb{R}^+ \times [0, 1] \times M$ and a smooth map $f : U \rightarrow \mathbb{R}^{2n}$ such that the following holds:*

i)

$$u(s, t) \in U_{s,t} := \{p \in M \mid (s, t, p) \in U\}$$

.

ii) *The mapping $f_{s,t} = f(s, t, \cdot) : U_{s,t} \rightarrow \mathbb{R}^{2n}$ is a diffeomorphism onto its image and it satisfies*

$$\begin{aligned} f_{s,i}(U_{s,i} \cap L_i) &\subset \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \\ df_{s,i}(q)J_i(q) &= J_{\text{std}} df_{s,i}(q), \quad q \in L_i \cap U_{s,i}, \quad i = 0, 1 \\ \partial_t f_{s,t}(p) + df_{s,t}(p)X_{H_t}(p) &= 0, \quad t \sim 0 \text{ and } t \sim 1, \quad p \in U_{s,t} \end{aligned} \quad (4.42)$$

iii) Besides, the neighborhoods $U_{s,t}$ and the mapping $f_{s,t}$ can be chosen to be s independent for large s .

Proof. The boundary conditions (4.12) include Hamiltonian vector field. We shall first use naturality to reduce this equation to an equation without the Hamiltonian term. Let ϕ_t be the Hamiltonian isotopy (4.2). Let

$$\begin{aligned}\tilde{u}(s, t) &= \phi_t^{-1}(u(s, t)), \quad \tilde{J}_t = (\phi_t)^* J_t, \\ \tilde{L}_i &= \phi_i^{-1}(L_i), \quad i = 0, 1\end{aligned}$$

Notice that $\tilde{u}(s, t)$ satisfies the following boundary conditions

$$\begin{aligned}\tilde{u}(s, i) &\in \tilde{L}_i, \quad i = 0, 1 \\ \tilde{J}_i(\tilde{u}(s, i))\partial_t \tilde{u}(s, i) &\in T_{\tilde{u}(s, i)}\tilde{L}_i, \quad i = 0, 1.\end{aligned}$$

We construct an open set $\tilde{U} \subset \mathbb{R}^+ \times [0, 1] \times M$ such that

$$\tilde{u}(s, t) \in \tilde{U}_{s,t} := \{p \in M \mid (s, t, p) \in \tilde{U}\}$$

and a smooth map $\tilde{f} : \tilde{U} \rightarrow \mathbb{R}^{2n}$ such that

$$\tilde{f}_{s,t}(\cdot) := \tilde{f}(s, t, \cdot) : \tilde{U}_{s,t} \rightarrow \mathbb{R}^{2n}$$

is a diffeomorphism onto its image and it satisfies the following properties

$$\begin{aligned}\tilde{f}_{s,i}(\tilde{U}_{s,i} \cap \tilde{L}_i) &\subset \mathbb{R}^n \times \{0\}, \\ (\tilde{f}_{s,i})_* \tilde{J}_{s,i}(x, 0) &= J_{\text{std}}, \quad (x, 0) \in \mathbb{R}^n \times \{0\} \cap \tilde{f}_{s,0}(\tilde{U}_{s,0}) \\ \partial_t \tilde{f}_{s,t}(p) &= 0, \quad \text{for } t \sim 0 \text{ and } t \sim 1.\end{aligned} \tag{4.43}$$

The open neighborhoods $\tilde{U}_{s,t}$ can be chosen to be s independent for large s as well as the mapping $\tilde{f}_{s,t}$. The construction of the maps $\tilde{f}_{s,t}$ satisfying the above properties is analogous to the construction of the map f_t in Lemma 4.2.3. For this reason we shall only sketch the construction of the maps $\tilde{f}_{s,t}$. We first construct a trivialization $e_{\tilde{u}}$ of the bundle \tilde{u}^*TM such that $e_{\tilde{u}}(s, t) : \mathbb{R}^{2n} \rightarrow T_{\tilde{u}(s,t)}M$ satisfies

$$\begin{aligned}e_{\tilde{u}}(s, i)(\mathbb{R}^n \times \{0\}) &= T_{\tilde{u}(s,i)}\tilde{L}_i, \quad i = 0, 1 \\ e_{\tilde{u}}(s, t) \circ J_{\text{std}} &= \tilde{J}_t(\tilde{u}(s, t)) \circ e_{\tilde{u}}(s, t).\end{aligned} \tag{4.44}$$

Notice that the smooth curve $\tilde{u}(s, t) = x(0)$, $s \geq s_0$. Construct as in Step 1 of Lemma 4.2.3, for a reference curve $\alpha(t) = x(0)$, a trivialization

$$e_{\tilde{u}}(s, t) : \mathbb{R}^{2n} \rightarrow T_{x(0)}M = T_{\tilde{u}(s,t)}M \text{ for } s \geq s_0.$$

This trivialization satisfies properties (4.44) for $s \geq s_0$ and is s independent. Next extend the trivialization by parallel transport along $\tilde{u}(s, t)$. More precisely we define

$$e_{\tilde{u}}(s, t)v := P_s(\tilde{u}(s, t), x(0))e_{\tilde{u}}(T_0, t)v,$$

where $P_s(\tilde{u}(s, t), x(0))$ denotes parallel transport in s direction along \tilde{u} from the point $x(0) = \tilde{u}(T_0, t)$ to the point $\tilde{u}(s, t)$. This parallel transport should be taken with respect to the connection $\tilde{\nabla} := \nabla - \frac{1}{2}\tilde{J}_t\nabla\tilde{J}_t$ and $\nabla = \nabla^t$ is a Levi-Civita connection of the metric h_t as in Lemma 2.3.5. Such parallel transport has the property

$$P_s(\tilde{u}(s, t), x(0))J_t(x(0))v = J_t(\tilde{u}(s, t))P_s(\tilde{u}(s, t), x(0))v, \quad \forall v.$$

The trivialization $e_{\tilde{u}}(s, t)$ obtained in this way satisfies the requirements (4.44).

Let $r_{s,t}$ be the injectivity radius of the metric h_t at the point $\tilde{u}(s, t)$. We define a mapping $\psi'_{s,t} : B_{r_{s,t}}(0) \rightarrow B_{r_{s,t}}(\tilde{u}(s, t)) = U'_{s,t}$ by

$$\psi'_{s,t}(\xi) = \exp_{\tilde{u}(s,t)}(e_{\tilde{u}}(s, t)\xi).$$

This mapping is obviously a diffeomorphism onto its image. Let $\phi'_{s,t}$ be its inverse. Let $J'_{s,t} = (\phi'_{s,t})_*\tilde{J}_t$. It follows from the properties (4.44) of the trivialization and metric g_t that

$$\begin{aligned} \phi'_{s,i}(\tilde{L}_i \cap U'_{s,t}) &\subset \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \\ J'_{s,t}(0) &= J_{std}. \end{aligned}$$

Analogously as in Lemma 4.2.3 in Step 3, we can make $J'_{s,t}$ standard on $\mathbb{R}^n \times \{0\}$ by composing with an appropriate isomorphism $\tilde{\Phi}_{s,t}$. Thus, the rest of the construction is completely analogous to the construction of the maps f_t in Lemma 4.2.3 and the mapping $f_{s,t}$ is given as a composition of $\phi'_{s,t}$ and $\tilde{\Phi}_{s,t}$. Finally the mapping $f_{s,t} : U_{s,t} = \phi_t(\tilde{U}_{s,t}) \rightarrow \mathbb{R}^{2n}$ is given by

$$f_{s,t}(p) = \tilde{f}_{s,t}(\phi_t^{-1}(p))$$

and it satisfies all the properties (4.42). \square

Definition 4.2.13. Let $u \in \mathcal{B}^+(x)$ and $f_{s,t} : U_{s,t} \rightarrow \mathbb{R}^{2n}$ be as in Lemma 4.2.12. Define $\mathcal{U}_u \subset \mathcal{B}^+(x)$ and $\Phi_u : \mathcal{U}_u \rightarrow W_{bc}^{2,2}(\mathbb{R}^+ \times [0, 1])$ by

$$\begin{aligned} \mathcal{U}_u &= \{v \in \mathcal{B}^+(x) | v(s, t) \in U_{s,t}\} \\ \Phi_u(v)(s, t) &= f_{s,t}(v(s, t)) \end{aligned}$$

Then $\Phi_u : \mathcal{U}_u \rightarrow W_{bc}^{2,2}(\mathbb{R}^+ \times [0, 1])$ is a coordinate chart on $\mathcal{B}^+(x)$. This defines a Hilbert manifold structure on $\mathcal{B}^+(x)$.

Proposition 4.2.14. *Let $\mathcal{B}^+(x)$ be defined as in 4.1.4 and let $\mathcal{P}^{3/2}(H, J)$ be as in 4.2.9. Observe the restriction to the non-Lagrangian boundary*

$$\mathcal{R} : \mathcal{B}^+(x) \rightarrow \mathcal{P}^{3/2}(H, J), \quad v \mapsto v(0, \cdot).$$

The mapping \mathcal{R} is a smooth surjective submersion.

Proof. Let $u \in \mathcal{B}^+(x)$, Φ_u and \mathcal{U}_u be as in Definition 4.2.13. Denote with α the smooth path $\alpha(t) = u(0, t)$. In a neighborhood \mathcal{U}_u the map \mathcal{R} is given by

$$\mathcal{R} = \Psi_\alpha \circ r \circ \Phi_u,$$

where Ψ_α is the inverse of the local chart Φ_α constructed in Lemma 4.2.10 and

$$r : W_{bc}^{2,2}(\mathbb{R}^+ \times [0, 1], \mathbb{R}^{2n}) \rightarrow H_{bc}^{3/2}$$

is a restriction map $r(\xi) = \xi(0, \cdot)$. As $d\mathcal{R} = d\Psi_\alpha \circ r \circ d\Phi_u$ and both $d\Psi_\alpha$ and $d\Phi_u$ are bijective it follows from Proposition 4.6.7 that \mathcal{R} is submersion. Surjectivity of the map \mathcal{R} follows again from Proposition 4.6.7. As the mapping r is surjective and Φ_u and Ψ_α are diffeomorphisms, it follows that the mapping \mathcal{R} maps a neighborhood of a smooth map $u \in \mathcal{B}^+(x)$ onto a neighborhood of $\alpha = u(0, \cdot)$. As the set of smooth $\alpha = u(0, t)$ is dense in $\mathcal{P}^{3/2}$ we have that the mapping \mathcal{R} is also surjective. \square

4.3 Proof of the Theorem 4.1.5

In this section we prove Theorem 4.1.5. The proof is based on the study of the vertical differential and main ingredients of the proof are already contained in the previous chapter.

4.3.1 (Vertical differential). Let $\mathcal{B} = \mathcal{B}^\pm(x)$ or $\mathcal{B} = \mathcal{B}^T$, where $\mathcal{B}^\pm(x)$ and \mathcal{B}^T are defined in 4.1.4. Let \mathcal{E} be a Hilbert space bundle over \mathcal{B} with the fiber over each $v \in \mathcal{B}$, $\mathcal{E}_v = W_{bc}^{1,2}(v^*TM)$, where

$$W_{bc}^{1,2}(v^*TM) = \{\xi \in W^{1,2}(v^*TM) \mid \xi(s, i) \in T_{v(s, i)}L_i, \ i = 0, 1\}.$$

Observe a section $\mathcal{S} : \mathcal{B} \rightarrow \mathcal{E}$ of this bundle given by

$$\mathcal{S}(v) = \bar{\partial}_{J_t, X_{H_t}} v = \partial_s v + J_t(v)(\partial_t v - X_{H_t}(v)).$$

Given $u \in \mathcal{S}^{-1}(0)$ denote by D_u the **vertical differential**

$$D_u = \pi_u \circ d\mathcal{S}(u) : T_u\mathcal{B} \rightarrow \mathcal{E}_u$$

where $\pi_u : T_{(u,0)}\mathcal{E} = T_u\mathcal{B} \oplus \mathcal{E}_u \rightarrow \mathcal{E}_u$ is the projection to the fiber. The vertical differential is given by

$$D_u(\hat{u}) = \nabla_s \hat{u} + J_t(u)(\nabla_t \hat{u} - \nabla_{\hat{u}} X_{H_t}(u)) + (\nabla_{\hat{u}} J_t(u))(\partial_t u - X_{H_t}(u)) \quad (4.45)$$

and it is independent of the choice of connection. Notice that if we take $\nabla = \nabla^t$ to be the Levi-Civita connection of the metric h_t which has the property that $L_i, i = 0, 1$ are totally geodesic with respect to h_i , then the tangent space $T_u\mathcal{B}$ can be described as follows

$$T_u\mathcal{B} := \left\{ \hat{u} \in W^{2,2}(u^*TM) \mid \begin{array}{l} \hat{u}(s, i) \in T_{u(s,i)}L_i, \quad i = 0, 1 \\ D_u(\hat{u}(s, i)) \in T_{u(s,i)}L_i, \quad i = 0, 1 \end{array} \right\}. \quad (4.46)$$

4.3.2 (Linearized operator at infinity). Let x be a solution of (4.3), we observe the vertical differential at x and we prove that it is symmetric. One can analogously as in Lemma 2.3 in [21] prove that it is bijective.

Theorem 4.3.3. *Denote with g_t a metric that we obtain by pairing J_t and ω .*

$$g_t(p)(v, w) = \omega_p(v, J_t(p)w) = \langle v, w \rangle_t$$

Let $x : [0, 1] \rightarrow M$ be a solution of (4.3). The operator

$$A : W_{bc}^{1,2}(x^*TM) \rightarrow L^2(x^*TM)$$

given by

$$A(\hat{x}) = J_t(x)(\nabla_t \hat{x} - \nabla_{\hat{x}} X_{H_t})$$

is symmetric with respect to the following scalar product

$$\langle \xi, \eta \rangle_{L^2} = \int_0^1 g_t(\xi(t), \eta(t)) dt = \int_0^1 \langle \xi(t), \eta(t) \rangle_t dt.$$

Proof. It is enough to prove that

$$\int_0^1 \langle \hat{y}, J_t(x)(\nabla_t \hat{x} - \nabla_{\hat{x}} X_{H_t}) \rangle_t = \int_0^1 \langle J_t(x)(\nabla_t \hat{y} - \nabla_{\hat{y}} X_{H_t}), \hat{x} \rangle_t$$

for all vector fields $\hat{x}, \hat{y} \in W_{bc}^{1,2}(x^*TM)$. Denote with $\dot{g}_t(p)(v, w) =$

$\omega(p)(v, \dot{J}_t(p)w)$, where $\dot{J}_t(p) = \frac{d}{dt}J_t(p)$.

$$\begin{aligned}
 I &= \int_0^1 \langle \hat{y}, J_t(x)(\nabla_t \hat{x} - \nabla_{\hat{x}} X_{H_t}) \rangle_t dt \\
 &= \int_0^1 \langle \hat{y}, J_t(x) \nabla_t \hat{x} \rangle_t dt - \int_0^1 \langle \hat{y}, J_t(x) \nabla_{\hat{x}} X_{H_t} \rangle_t dt \\
 &= - \int_0^1 \langle J_t(x) \hat{y}, \nabla_t \hat{x} \rangle_t dt - \int_0^1 \langle \hat{y}, J_t(x) \nabla_{\hat{x}} X_{H_t} \rangle_t dt \\
 &= \int_0^1 \langle \nabla_t (J_t(x) \hat{y}), \hat{x} \rangle_t dt - \overbrace{\langle J_1(\hat{y}(1)), \hat{x}(1) \rangle_1}^{bc=0} \\
 &\quad + \overbrace{\langle J_0(\hat{y}(0)), \hat{x}(0) \rangle_0}^{bc=0} + \int_0^1 \dot{g}_t(J_t(x) \hat{y}, \hat{x}) dt - \int_0^1 \langle \hat{y}, J_t(x) \nabla_{\hat{x}} X_{H_t} \rangle_t dt \\
 &= \int_0^1 \langle \dot{J}_t(x) \hat{y}, \hat{x} \rangle_t dt + \int_0^1 \langle (\nabla_{\dot{x}} J_t) \hat{y}, \hat{x} \rangle_t dt + \int_0^1 \langle J_t(x) \nabla_t \hat{y}, \hat{x} \rangle_t dt \\
 &\quad + \int_0^1 \dot{g}_t(J_t(x) \hat{y}, \hat{x}) dt - \int_0^1 \langle \hat{y}, J_t(x) \nabla_{\hat{x}} X_{H_t} \rangle_t dt
 \end{aligned}$$

Notice that the first and the fourth term of the previous equality cancel out. This follows by differentiating the sum $0 = \frac{d}{dt}(\omega(w, J_t(p)v) + \omega(v, J_t(p)w))$. Thus we have

$$\begin{aligned}
 I &= \int_0^1 \langle (\nabla_{\dot{x}} J_t) \hat{y}, \hat{x} \rangle_t dt + \overbrace{\int_0^1 \langle J_t(x)(\nabla_t \hat{y} - \nabla_{\hat{y}} X_{H_t}), \hat{x} \rangle_t dt}^J \\
 &\quad + \int_0^1 \langle J_t(x) \nabla_{\hat{y}} X_{H_t}, \hat{x} \rangle_t dt - \int_0^1 \langle \hat{y}, J_t \nabla_{\hat{x}} X_{H_t} \rangle_t dt
 \end{aligned}$$

Write

$$\int_0^1 \langle J_t(x) \nabla_{\hat{y}} X_{H_t}, \hat{x} \rangle_t dt = \int_0^1 \langle \nabla_{\hat{y}} (J_t X_H), \hat{x} \rangle_t dt - \int_0^1 \langle (\nabla_{\hat{y}} J_t) X_{H_t}, \hat{x} \rangle_t dt,$$

and similarly

$$\int_0^1 \langle \hat{y}, J_t \nabla_{\hat{x}} X_{H_t} \rangle_t dt = \int_0^1 \langle \hat{y}, \nabla_{\hat{x}} (J_t X_H) \rangle_t dt - \int_0^1 \langle \hat{y}, (\nabla_{\hat{x}} J_t) X_{H_t} \rangle_t dt.$$

As $J_t X_{H_t} = \nabla H$, we have that

$$\langle \nabla_{\hat{y}} \nabla H, \hat{x} \rangle_t = \langle \hat{y}, \nabla_{\hat{x}} \nabla H \rangle_t.$$

Thus it follows that

$$I = J + \int_0^1 \langle (\nabla_{\hat{x}} J_t) \hat{y}, \hat{x} \rangle_t dt - \int_0^1 \langle (\nabla_{\hat{y}} J_t) X_{H_t}, \hat{x} \rangle_t + \int_0^1 \langle \hat{y}, (\nabla_{\hat{x}} J_t) X_{H_t} \rangle_t dt$$

As $\nabla_{\hat{y}} J_t$ is skew symmetric it follows that

$$- \int_0^1 \langle (\nabla_{\hat{y}} J_t) \dot{x}, \hat{x} \rangle_t = \int_0^1 \langle (\nabla_{\hat{y}} J_t) \hat{x}, \dot{x} \rangle_t.$$

Now the sum

$$\int_0^1 \langle (\nabla_{\hat{x}} J_t) \hat{y}, \hat{x} \rangle_t dt + \int_0^1 \langle (\nabla_{\hat{y}} J_t) \hat{x}, \dot{x} \rangle_t + \int_0^1 \langle (\nabla_{\hat{x}} J_t) \dot{x}, \hat{y} \rangle_t dt \quad (4.47)$$

is equal to zero and this follows from the compatibility of ω and J_t . For more details we refer to the Appendix in [16]. \square

Lemma 4.3.4 (Unitary trivialization). *Let $u \in \mathcal{B}^+(x)$ be smooth such that $u(s, t) = x(t)$, $s \geq s_0$. There exists an open set $U \subset \mathbb{R}^+ \times [0, 1] \times M$ such that*

$$u(s, t) \in U_{s,t} = \{p \in M \mid (s, t, p) \in V\}$$

and a smooth map $\Phi : U \times \mathbb{R}^{2n} \rightarrow TM$ such that $\Phi_{s,t} = \Phi(s, t, \cdot)$ has the following properties:

i) $\Phi_{s,t}(p) : \mathbb{R}^{2n} \rightarrow T_p M$, $p \in U_{s,t}$ is a vector space isomorphism.

ii) $\Phi_{s,t}(p)$ is complex, i.e.

$$\Phi_{s,t}(p) J_0 = J_t(p) \Phi_{s,t}(p), \quad p \in U_{s,t}.$$

iii) $\Phi_{s,i}(q) : \mathbb{R}^n \times \{0\} \rightarrow T_q L_i$, $q \in L_i \cap U_{s,i}$, $i = 0, 1$

iv) The mapping $\Phi_{s,t}$ is s independent for s sufficiently large, thus $\Phi_{s,t} = \Phi_t$ for s sufficiently large.

v) $\Phi_t(x(t))$ is symplectic

$$\omega(\Phi_t(x(t)) \cdot, \Phi_t(x(t)) \cdot) = \omega_0(\cdot, \cdot).$$

Proof. In steps 1-4 we assume that the Hamiltonian term vanishes, thus $u(s, t) = p_0$ for $s \geq s_0$. In the last Step we reduce the case $H \neq 0$ to the case $H = 0$ by naturality.

Step 1. Construction of the trivialization $\Phi_\infty(t)$ of $T_{p_0} M$ satisfying

the properties $i) - v)$.

This trivialization can be constructed analogously as the trivialization e_α in the Step 1 of the proof of Lemma 4.2.3, or in the following way. There exists a smooth family of Lagrangian planes $T_{p_0}L_t$ connecting $T_{p_0}L_0$ and $T_{p_0}L_1$. Let $e_i(t) \in T_{p_0}L_t$, $i = 1, \dots, n$ be orthonormal frame with respect the metric $\omega(\cdot, J_t(p_0)\cdot)$. Observe the unitary frame $\{e_i(t)J_t(p_0)e_i(t)\}_{i=1, \dots, n}$. Let $\{v_i, J_0v_i\}_{i=1, n}$ be the standard basis of \mathbb{R}^{2n} (v_i is the orthonormal basis of $\mathbb{R}^n \times \{0\}$). We define the trivialization

$$\Phi_\infty : [0, 1] \times \mathbb{R}^{2n} \rightarrow T_{p_0}M$$

by

$$\Phi_\infty(t)v_i = e_i(t), \quad \Phi_\infty(t)J_0v_i = J_t(p_0)e_i(t).$$

Notice that the the constructed trivialization maps $\mathbb{R}^n \times \{0\}$ to $T_{p_0}L_i$, $i = 0, 1$ and that is unitary.

Step 2. Extension of the trivialization $\Phi_\infty(t)$ to the neighborhood of p_0 .

There exists an open set $V \subset I \times M$ such that

$$p_0 \in V_t = \{p \in M | (t, p) \in V\},$$

and a smooth map $\tilde{\Phi} : V \times \mathbb{R}^{2n} \rightarrow TM$ such that $\tilde{\Phi}_t := \tilde{\Phi}(t, \cdot)$ satisfies the following

- 1) $\tilde{\Phi}_t(p) : \mathbb{R}^{2n} \rightarrow T_pM$, $p \in V_t$ is a vector space isomorphism for all $p \in V_t$.
- 2) $\tilde{\Phi}_t(p)J_0 = J_t(p)\tilde{\Phi}_t(p)$ for all $p \in V_t$.
- 3) $\tilde{\Phi}_i(p) : \mathbb{R}^n \times \{0\} \rightarrow T_pL_i$ for all $p \in L_i \cap V_i$, $i = 0, 1$.
- 4) $\tilde{\Phi}_t(x(t)) = \Phi_\infty(t)$.

Let g_t be a family of metrics as in Lemma 2.3.5 Let ∇^t be a Levi-Civita connection of the metric g_t and let $\tilde{\nabla}^t$ be a complex linear connection associated to ∇^t

$$\tilde{\nabla}_\lambda^t v = \nabla_\lambda^t v - \frac{1}{2}J_t(\nabla_\lambda^t J_t)v.$$

For a point p in a geodesic neighborhood of p_0 we define the trivialization $\tilde{\Phi}_t(p)$ as follows

$$\tilde{\Phi}_t(p)v := P_\gamma(p_0, p)\Phi_\infty(t)v,$$

where P_γ denotes parallel transport (with respect to $\tilde{\nabla}^t$) along geodesic $\gamma(\lambda)$ connecting p_0 and p . As P_γ commutes with J_t it follows that $\tilde{\Phi}_t$ satisfies 2). As L_i , $i = 0, 1$ are totally geodesic with respect to $\tilde{\nabla}^i$, we have that 3) is also satisfied. We can assume that $U_{s,t} := V_t$, $s \geq s_0$.

Step 3. Extension of the trivialization $\Phi_\infty(t)$ to the trivialization of u^*TM .

Use parallel transport with respect to $\tilde{\nabla}^t$, defined as in Step 2, along $u(s, t)$ in the direction of s . In this way we construct a smooth mapping

$$\Phi : \mathbb{R}^+ \times [0, 1] \times \mathbb{R}^{2n} \rightarrow u^*TM,$$

or equivalently we construct the trivialization

$$\Phi_{s,t}(u(s, t)) = \Phi(s, t, u(s, t)) : \mathbb{R}^{2n} \rightarrow T_{u(s,t)}M.$$

defined by

$$\Phi_{s,t}(u(s, t)) := P_u(u(s_0, \cdot), u(s, \cdot))\Phi_\infty(t)v,$$

where $P_u(u(s_0, \cdot), u(s, \cdot))$ denotes parallel transport along u .

Step 4. Extension of the trivialization $\Phi_{s,t}(u(s, t))$ to some neighborhoods $U_{s,t}$ of $u(s, t)$.

This can be done analogously as in Step 2, using parallel transport along geodesics. Thus the neighborhoods $U_{s,t}$ are just geodesic neighborhoods of $u(s, t)$.

Step 5. Reducing the general case to the case $H = 0$.

Let $\tilde{u}(s, t) = \phi_t^{-1}(u(s, t))$ and $\tilde{J}_t = \phi_t^*J_t$, where Φ_t is Hamiltonian isotopy (4.2). Then $\tilde{u}(s, t) = p_0$, $s \geq s_0$ and we can construct as in Steps 1-4 the trivialization $\tilde{\Phi}$ satisfying the properties $i) - v)$ applied to the curve \tilde{u} and the almost complex structure \tilde{J}_t . Then the mapping Φ defined as follows

$$\Phi_{s,t}(p) := d\phi_t(\phi_t^{-1}(p))\tilde{\Phi}_{s,t}(\phi_t^{-1}(p))$$

satisfies the properties $i) - v)$.

□

4.3.5 (Conjugate operator). Let u be a solution of the equation (4.5) and suppose that u converges exponentially toward $x \in \mathcal{C}(L_0, L_1; H)$. Notice that u is just of $W^{2,2}$ class, though it is smooth away from the non-Lagrangian boundary. Let D_u be the vertical differential as in 4.3.1 and let $\Phi_{s,t}$ be as in

Lemma 4.3.4. Without loss of generality we can assume that $u(s, t) \in U_{s,t}$, where $U_{s,t}$ are as in Lemma 4.3.4. We abbreviate $\Phi = \Phi_{s,t}(u(s, t))$. It follows from the properties of the trivialization Φ that the conjugate operator $D = \Phi^{-1} \circ D_u \circ \Phi$ has the following form

$$D\xi = \Phi^{-1} D_u \circ (\Phi\xi) = \partial_s \xi + J_0 \partial_t \xi + S(s, t)\xi = \partial_s \xi + A(s)\xi \quad (4.48)$$

The matrix valued function $S \in W^{1,2}(\mathbb{R}^+ \times [0, 1], \mathbb{R}^{2n \times 2n})$ is given by

$$S(v) = \overbrace{\Phi^{-1} \nabla_s (\Phi v)}^{C(v)} + J_0 \overbrace{\Phi^{-1} (\nabla_t (\Phi v) - \nabla_{\Phi(v)} X_{H_t})}^{B(v)} + \overbrace{\Phi^{-1} (\nabla_{\Phi(v)} J) (\partial_t u - X_{H_t})}^{E(v)}$$

Define a smooth matrix valued function $B_\infty : [0, 1] \rightarrow \mathbb{R}^{2n \times 2n}$ by

$$B_\infty(t)v = \Phi_t(x(t))^{-1} \left(\nabla_t (\Phi_t(x(t))v) - \nabla_{\Phi_t(x(t))v} X_{H_t}(x(t)) \right),$$

where Φ_t is given as in part *iv*) of Lemma 4.3.4. From exponential decay of u it follows that the C, E converge exponentially to zero and the function B converges toward B_∞ . One can also see that $J_0 B_\infty(t)$ is symmetric for all t . Besides the matrix valued functions $C(s, i), E(s, i)$ and $J_0(B(s, i) - B_\infty(i))$ map $\mathbb{R}^n \times \{0\}$ into itself for $i = 0, 1$. These boundary properties will imply that if we observe the operators $A(s) = J_0 \partial_t + S(s, t)$ and we fix $H^1 = \text{Dom}(A(s))$, as in (3.2), then also $H^2 = \text{Dom}(A(s)^2)$ will be independent of s .

Corollary 4.3.6. *Let H^i , $i = 0, 1$ be as in (3.2) and let C, B, E and B_∞ be as in 4.3.5. Then for any $k \in \mathbb{N}$ we have*

$$\begin{aligned} \lim_{s \rightarrow \infty} \|C\|_{C^k([s, +\infty) \times [0, 1])} &= 0, \quad \lim_{s \rightarrow \infty} \|E\|_{C^k([s, +\infty) \times [0, 1])} = 0 \\ \lim_{s \rightarrow \infty} \|B - B_\infty\|_{C^k([s, +\infty) \times [0, 1])} &= 0. \end{aligned}$$

Moreover

- i) The operator $J_0 \partial_t + J_0 B_\infty : H^1 \rightarrow H^0$ is bijective and self-adjoint.*
- ii) The functions $C(s, i), E(s, i)$ and $J_0(B(s, i) - B_\infty(i))$ map $\mathbb{R}^n \times 0$ to itself for $i = 0, 1$.*

Proof. The fact that C, E and the difference $B - B_\infty$ converge to zero, follows from the fact that u converges exponentially to $x(t)$. Remember that ∇ in the definition of the matrix valued functions B, C and E is the Levi-Civita connection of the metric g_t such that L_i are totally geodesic with respect to g_i , $i = 0, 1$. This implies that C and E satisfy *ii*). To prove that

$J_0(B(s, i) - B_\infty(i)) : \mathbb{R}^n \times \{0\} \rightarrow \mathbb{R}^n \times \{0\}$ it is enough to prove that $\partial_s B(s, i) : \mathbb{R}^n \times \{0\} \rightarrow \{0\} \times \mathbb{R}^n$. We prove this fact in the case that Hamiltonian term vanishes. Notice that the trivialization was constructed such that the general case $H \neq 0$ can be reduced to the case $H = 0$. The following equalities hold

$$\begin{aligned} \nabla_s(\Phi_{s,t}(u)B(s, t)v) &= \nabla_s \nabla_t(\Phi_{s,t}(u)v) \\ \nabla_s(\Phi_{s,t}(u))B(s, t)v + \Phi_{s,t}(u)\partial_s B(s, t)v &= \nabla_t \nabla_s(\Phi_{s,t}(u)v) + R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v \end{aligned} \quad (4.49)$$

Where R denotes the Riemann curvature tensor. Notice that the first term on both left and right side of the upper equality vanishes, as $\Phi_{s,t}(u) = P_s \circ \Phi_t(x(t))$, where P_s denotes parallel transport along u in the direction of s . Thus we have

$$\Phi_{s,t}(u)\partial_s B(s, t)v = R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v.$$

It is left to prove that $R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v|_{t=0} \perp T_{u(s,0)}L_0$ and analogously for $t = 1$. Thus it is enough to prove that

$$\langle R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v|_{t=0}, w \rangle_0 = 0, \quad \forall w \in T_{u(s,0)}L_0$$

From the properties of the curvature R we have

$$\langle R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v, w \rangle_0 = \langle R(\Phi_{s,t}(u)v, w)\partial_s u, \partial_t u \rangle. \quad (4.50)$$

As $R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z$ and all three vector fields $\Phi_{s,t}(u)v|_{t=0}$, w and $\partial_s u$ are tangent to L_0 and as L_i are totally geodesic for g_i we have that $R(\Phi_{s,t}(u)v, w)\partial_s u$ is also tangent to L_0 . As $\partial_t u$ is orthogonal to $T_{u(s,0)}L_0$ we have that $\langle R(\partial_s u, \partial_t u)\Phi_{s,t}(u)v, w \rangle_0 = 0$ for all $w \in T_{u(s,0)}L_0$. \square

Theorem 4.3.7. *Let \mathcal{B} and $u \in \mathcal{B}$ be as in 4.3.1 and let $D_u : T_u \mathcal{B} \rightarrow \mathcal{E}_u$ be vertical differential as in 4.3.1. Then the following holds*

- a) *The operator D_u is surjective.*
- b) *Suppose that $\mathcal{B} = \mathcal{B}^+$. Then there exists a constant $c > 0$ such that the following inequality holds for all $\hat{u} \in T_u \mathcal{B}$*

$$\|\hat{u}\|_{2,2} \leq c \left(\|D_u \hat{u}\|_{1,2} + \|\hat{u}(0, \cdot)\|_{3/2} \right) \quad (4.51)$$

Analogous results holds in the case $\mathcal{B} = \mathcal{B}^T$. For every $T > 0$ there exists a constant c such that the following inequality holds for all $\hat{u} \in T_u \mathcal{B}^T$.

$$\|\hat{u}\|_{2,2} \leq c \left(\|D_u(\hat{u})\|_{1,2} + \|\hat{u}(-T, \cdot)\|_{3/2} + \|\hat{u}(T, \cdot)\|_{3/2} \right) \quad (4.52)$$

Proof. a) Let Φ be the trivialization of u^*TM as in Lemma 4.3.4. Let D be the conjugate operator as in (4.48), $D = \Phi^{-1} \circ D_u \circ \Phi$. Conjugation by Φ identifies the tangent space $T_u\mathcal{B}$ with the Hilbert space $H_{bc}^2(I \times [0, 1])$ given by

$$\begin{aligned} H_{bc}^2(I \times [0, 1]) &= \left\{ \xi \in W^{2,2}(I \times [0, 1]) \mid \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \\ D\xi(s, i) \in \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \end{array} \right\} \\ &= W^{2,2}(I, L^2([0, 1])) \cap L^2(I, \text{Dom}(A(s))^2) \end{aligned}$$

notice that from 4.3.5 and 4.3.6 we have that $\text{Dom}(A(s)^2)$ is s independent. It follows from Corollary 3.3.7 that D_u is surjective.

b) Let $\hat{u}(s, t) = \Phi_{s,t}(u(s, t))\xi(s, t)$, where Φ is the trivialization constructed in Lemma 4.3.4 and let $D = \Phi^{-1} \circ D_u \circ \Phi$ be as in (4.48). As the operator D has the form (3.37), the conclusion of Corollary 3.3.6 holds. Thus the inequality (4.51) follows from Corollary 3.3.6 part b), whereas the inequality (4.52) follows from the same Corollary part a). \square

Proof of Theorem 4.1.5. This is just an easy corollary of the previous Theorem. a) We prove that $\mathcal{M}^+ \subset \mathcal{B}^+$ is a smooth Hilbert submanifold. The proof that $\mathcal{M}^T \subset \mathcal{B}^T$ is a smooth submanifold is analogous. Let $u \in \mathcal{M}^+$ and let D_u be the vertical differential as in 4.3.1. It is enough to prove that $D_u : T_u\mathcal{B} \rightarrow \mathcal{E}_u$ is surjective, as this implies that the section $\mathcal{S} = \bar{\partial}_{J_t, X_t}$ is transverse to the 0-section and hence the set $\mathcal{M}^+ = \mathcal{S}^{-1}(0)$ is a smooth submanifold. This follows from Theorem 4.3.7 part a).

b) To prove that the maps $i^\pm : \mathcal{M}^\pm \rightarrow \mathcal{P}^{3/2}$ are immersions notice that $T_u\mathcal{M}^\pm \cong \text{Ker}(D_u)$, thus it follows directly from the inequality (4.51) that the maps $di^\pm(u)$ are injective. In the case of finite strips it follows from the inequality (4.52) that the mapping i^T is an immersion. That the maps i^\pm and i^T are also injective follows from the unique continuation of holomorphic curves. \square

4.4 Embedding into the path space

In this section we prove Theorem 4.1.7.

4.4.1 (Zero Hamiltonian). Let $x \in \mathcal{C}(L_0, L_1, H)$ and let $\mathcal{M}^\pm(x; H, J)$ and $\mathcal{M}^T(H, J)$ be defined as in 4.1.4. We have proved in Theorem 4.1.5 that these moduli spaces are Hilbert manifolds and that they can be immersed into the Hilbert manifold of paths $\mathcal{P}^{3/2}(H, J)$. In Remark 4.1.9 we have

explained that it is enough to study these manifolds in the case that the Hamiltonian function $H = 0$. Let \tilde{J} , \tilde{x} and \tilde{L}_i , $i = 0, 1$ be as in Remark 4.1.9. Notice that it follows from 4.3.2 that the linearized operator

$$A : W_{bc}^{1,2}(\tilde{x}^*TM) \rightarrow L^2(\tilde{x}^*TM), \quad A(\hat{x}) := \tilde{J}_t(\tilde{x})\partial_t\hat{x}$$

is bijective and self-adjoint. This will be crucial for all the proofs. We abbreviate $\mathcal{M}^\infty = \mathcal{M}^\infty(\tilde{x}; 0, \tilde{J})$, $\mathcal{M}^T = \mathcal{M}^T(0, \tilde{J})$ and $\mathcal{P}^{3/2} = \mathcal{P}^{3/2}(0, \tilde{J})$. Here we shall consider only those curves that are close enough to the constant curve

$$p := \tilde{x} \in \tilde{L}_0 \cap \tilde{L}_1$$

and have sufficiently small energy. We first explain when a path $\alpha \in \mathcal{P}^{3/2}$ is in an ϵ neighborhood of a constant path p . Then we introduce subsets $\mathcal{M}_\epsilon^\infty$ and \mathcal{M}_ϵ^T of the moduli space of holomorphic curves \mathcal{M}^∞ and \mathcal{M}^T and we prove that they can be embedded, by taking the restriction to the boundary, into the Hilbert manifold of paths $\mathcal{P}^{3/2}$. To simplify the notation we omit \sim .

4.4.2 (ϵ -neighborhood of $p \in L_0 \cap L_1$). We assume that the Hamiltonian is 0. Local chart in the neighborhood of p within the Hilbert manifold $\mathcal{P}^{3/2}$ is given as in Lemma 4.2.10. Remember that the local chart in the neighborhood of a constant path $p \in \mathcal{P}^{3/2}$, $\Phi_p : \mathcal{U}_p \rightarrow \mathcal{W}_p$, is given by $\Phi_p(\gamma) = f_t(\gamma)$, where $f_t : U_p \rightarrow \mathbb{R}^{2n}$ is a smooth family of maps, $U_p \subset M$ is an open neighborhood of p that doesn't contain other intersection points of $L_0 \cap L_1$ and f_t has the following additional properties:

- 1) $f_t : U_p \rightarrow f_t(U_p) \subset \mathbb{R}^{2n}$ is a diffeomorphism for all t and $f_t(p) = 0$ for all t .
- 2) $f_i(L_i \cap U_p) = (\mathbb{R}^n \times \{0\}) \cap f_i(U_p)$, $i = 0, 1$, and $(\partial_t f_t)|_{t=0,1} = 0$.
- 3) If $\tilde{J}_t = (f_t)_* J_t$, then $\tilde{J}_t(x, 0) = J_{std}$ for all $t \in [0, 1]$ and for all $(x, 0) \in (\mathbb{R}^n \times \{0\}) \cap f_t(U_p)$.

We say that a curve $\gamma \in \mathcal{P}^{3/2}$ is in the ϵ neighborhood of $p \in \mathcal{P}^{3/2}$ and we write $\gamma \in \mathcal{U}_\epsilon(p)$ iff $\gamma \subset U_p$ and $\xi(t) := f_t(\gamma(t)) = \Phi_p(\gamma)(t) \subset H^{3/2}$ satisfies

$$\|\xi\|_{3/2} < \epsilon.$$

Notice that it makes sense to define ϵ -neighborhood of a constant path p only in the case that $\epsilon > 0$ is sufficiently small.

We define analogously an ϵ -neighborhood of a constant strip $p \in \mathcal{B}^\pm$ (or

$p \in \mathcal{B}^T$). A strip $u \in \mathcal{B}^\pm$ is in an ε -neighborhood of p if $\text{Im}(u) \subset U_p$ and $\xi(s, t) = f_t(u(s, t))$ satisfies

$$\|\xi\|_{2,2} < \epsilon,$$

where f_t is a local chart as above.

4.4.3 (Monotonicity). Let U_p be the neighborhood of a point p as in Remark 4.4.2. We shall be interested only in those holomorphic curves that are contained in this neighborhood. In Theorems 2.1.3 and 2.1.5, we prove that the energy of a holomorphic curve $u : I \times [0, 1] \rightarrow N$ and $\sup_t \sup_{s \in \partial I} d(u(s, t), p)$ control the distance $\sup_{s,t} d(u(s, t), p)$. Let \hbar and $\epsilon_0 > 0$ be such that each holomorphic curve u which satisfies

$$E(u) < \hbar, \quad u|_{\partial I \times [0,1]} \in \mathcal{U}_{\epsilon_0}(p)$$

is contained in U_p , i.e.

$$u(s, t) \in U_p, \quad \forall (s, t) \in I \times [0, 1].$$

Here $\mathcal{U}_{\epsilon_0}(p)$ denotes the ϵ_0 neighborhood of a constant path p in the Hilbert manifold $\mathcal{P}^{3/2}$ as in definition 4.4.2.

Definition 4.4.4. Let ϵ_0 and \hbar be as in 4.4.3 and let \mathcal{M}^T and \mathcal{M}^∞ be as in 4.4.1. For $\epsilon \leq \epsilon_0$ we define the following subsets of the moduli spaces of holomorphic strips:

$$\begin{aligned} \mathcal{M}_\epsilon^\infty &= \left\{ (u^+, u^-) \in \mathcal{M}^\infty \mid u^\pm(0, \cdot) \in \mathcal{U}_\epsilon(p), \quad E(u^\pm) < \hbar \right\} \\ \mathcal{M}_\epsilon^T &= \left\{ u \in \mathcal{M}^T \mid u(\pm T, \cdot) \in \mathcal{U}_\epsilon(p), \quad E(u) < \hbar \right\}. \end{aligned}$$

Here \mathcal{U}_ϵ denotes ϵ -neighborhood of a constant path p in the Hilbert manifold of paths $\mathcal{P}^{3/2}$ as in 4.4.2.

4.4.5 (Local setup). Holomorphic curves $u \in \mathcal{M}_\epsilon^T$ ($(u^+, u^-) \in \mathcal{M}_\epsilon^\infty$) are contained in the small neighborhood U_p of the point $p \in M$, as explained in 4.4.3. Thus instead of observing holomorphic curves on M we can work in \mathbb{R}^{2n} . We reformulate the setup in local coordinates.

Let $v = \Phi_p(u) = f_t(u)$ be the image of a holomorphic curve u contained in U_p . Here f_t is as in 4.4.2. As u is a J_t holomorphic curve it follows that $v = f_t(u)$ satisfies the following equation

$$\bar{\partial}_{\tilde{J}, \tilde{X}} v = \partial_s v + \tilde{J}_t(v)(\partial_t v - \tilde{X}_t(v)) = 0, \quad (4.53)$$

where \tilde{X}_t and \tilde{J}_t have the following properties.

1) \tilde{X}_t is a smooth vector field given by

$$\tilde{X}_t(x) = \partial_t f_t(f_t^{-1}(x)).$$

2) $\tilde{X}_t(x) = 0$ for $t = 0, 1$ and for all $x \in \mathbb{R}^{2n}$.

3) $\tilde{X}_t(0) = 0$ for all $t \in [0, 1]$.

4) $\tilde{J}_t = (f_t)_* J_t$ is a smooth family of almost complex structures with the properties

$$\tilde{J}_t(x, 0) = J_{\text{std}}, \quad t = 0, 1, \quad x \in \mathbb{R}^n. \quad (4.54)$$

New boundary data are given by

$$v(s, 0) \in \tilde{L}_0 = f_0(L_0) \subset \mathbb{R}^n \times \{0\}, \quad v(s, 1) \in \tilde{L}_1 = f_1(L_1) \subset \mathbb{R}^n \times \{0\} \quad (4.55)$$

Hence it follows that $v \in H_{bc}^2(I \times [0, 1], \mathbb{R}^{2n})$, where $H_{bc}^2(I \times [0, 1])$ is given by (3.11). The condition that the intersection of Lagrangian submanifolds $L_0 \cap L_1$ is transverse translates into the following condition in \mathbb{R}^{2n} . Let $\tilde{\psi}_t$ be the flow of the vector field \tilde{X}_t

$$\partial_t \tilde{\psi}_t(x) = \tilde{X}_t(\tilde{\psi}_t(x)), \quad \tilde{\psi}_0 = \mathbb{1}.$$

Notice that $\tilde{\psi}_t = f_t \circ f_0^{-1}$. Then

$$L_0 \pitchfork L_1 \quad \text{if and only if} \quad \tilde{\psi}_1(\tilde{L}_0) \pitchfork \tilde{L}_1.$$

4.4.6 (Linearized operator). Let $\bar{\partial}_{\tilde{J}, \tilde{X}}$ be as in (4.53) and let D_0 be its linearization at the origin.

$$D_0 \xi = \partial_s \xi + \tilde{J}_t(0)(\partial_t \xi - d\tilde{X}_t(0)\xi) = \partial_s \xi + A\xi$$

Then the linear operator

$$\begin{aligned} A &: H_{bc}^1([0, 1], \mathbb{R}^{2n}) \rightarrow L^2([0, 1], \mathbb{R}^{2n}), \\ A &= \tilde{J}_t(0)(\partial_t - d\tilde{X}_t(0)) \end{aligned}$$

is bijective and self-adjoint, where

$$H_{bc}^1([0, 1]) = \left\{ \xi \in H^1([0, 1], \mathbb{R}^{2n}) \left| \xi(i) \in \mathbb{R}^n \times \{0\}, \quad i = 0, 1 \right. \right\}.$$

The operator A is conjugate via $df_t(p)$ to the operator $B = J_t(p)\partial_t$,

$$A\xi = df_t(p)J_t(p)\partial_t(df_t(p)^{-1}\xi).$$

As the operator $B = J_t(p)\partial_t$ is bijective and self-adjoint it follows that the operator A is also bijective and self-adjoint with respect to the appropriately chosen metric. More precisely, let $\xi, \eta \in L^2([0, 1], \mathbb{R}^{2n})$ we define the L^2 scalar product by

$$\langle \xi, \eta \rangle := \int_0^1 \omega(df_t(p)^{-1}\xi(t), J_t(p)df_t(p)^{-1}\eta(t))dt. \quad (4.56)$$

Remember that ω and J_t are compatible. For $\xi, \eta \in H_{bc}^1([0, 1])$, we have

$$\begin{aligned} \langle \xi, A\eta \rangle &= \int_0^1 \omega(df_t(p)^{-1}\xi(t), -\partial_t(df_t(p)^{-1}\eta(t)))dt \\ &= \int_0^1 \omega(\partial_t(df_t(p)^{-1}\xi(t)), df_t(p)^{-1}\eta(t))dt \\ &= \int_0^1 \omega(-J_t(p)df_t(p)^{-1}(A\xi), df_t(p)^{-1}\eta(t))dt \\ &= \int_0^1 \omega(df_t(p)^{-1}\eta(t), J_t(p)df_t(p)^{-1}(A\xi))dt \\ &= \langle \eta, A\xi \rangle. \end{aligned}$$

The second equality follows by partial integration using the boundary conditions, i.e. $df_i(p) : T_p L_i \rightarrow \mathbb{R}^n \times \{0\}$, $i = 0, 1$. Injectivity of the operator A is easy to prove. Notice the following

$$A\xi = 0 \Leftrightarrow \zeta(t) = df_t(p)^{-1}\xi(t) = \text{const.} \in T_p M$$

As $\zeta(0) \in T_p L_0$ and $\zeta(1) \in T_p L_1$ and the intersection $T_p L_0 \cap T_p L_1 = \{0\}$, it follows that $\zeta(t) = 0$ and thus $\xi \equiv 0$.

To simplify the notation we shall write J_t and X_t instead of \tilde{X}_t and \tilde{X}_t further on. Let v be a solution of the equation (4.53) and let D_v be the linearization of the same equation, then

$$\begin{aligned} D_v \xi &= \partial_s \xi + J_t(v)(\partial_t \xi - dX_t(v)\xi) + (dJ_t(v)\xi)(\partial_t v - X_t(v)). \\ &= \partial_s \xi + J_t(v)(\partial_t \xi - dX_t(v)\xi) + (dJ_t(v)\xi)J_t(v)\partial_s v. \end{aligned} \quad (4.57)$$

4.4.7 (Proof of Theorem 4.1.7). Let $\mathcal{M}_\epsilon^\infty$ and \mathcal{M}_ϵ^T be defined as in 4.4.4. In order to prove Theorem 4.1.7 it is enough to prove that for $\epsilon > 0$ sufficiently small the manifolds $\mathcal{M}_\epsilon^\infty$ and \mathcal{M}_ϵ^T embed into path space by

taking the restriction to the boundary. In Theorem 4.1.5 we have proved that the maps $i^\infty = i^+ \times i^-$ and i^T are injective immersions. Locally each immersion is an embedding. There exists $r_1 > 0$ such that the restriction of the map i^\pm (analogously i^T) to the $\mathcal{U}_{r_1}(p)$ is an embedding, where $\mathcal{U}_{r_1}(p) \subset \mathcal{B}^\pm$ (or $\mathcal{U}_{r_1}(p) \subset \mathcal{B}^T$) is r_1 neighborhood of p as in 4.4.2 . Now the proof for infinite strips follows directly from the Theorem 4.4.8 and for finite strips it follows from Remark 4.4.9. More precisely it follows from Theorem 4.4.8 that for $\epsilon > 0$ sufficiently small each $(u^+, u^-) \in \mathcal{M}_\epsilon^\infty$ satisfies

$$\|f_t(u^\pm)\|_{2,2} < r_1$$

Thus $u^+ \in \mathcal{U}_{r_1}(p) \subset \mathcal{B}^+$ and $u^- \in \mathcal{U}_{r_1}(p) \subset \mathcal{B}^-$. \square

Theorem 4.4.8. *There exist $\epsilon > 0$ and $c > 0$ with the following significance. Let u be an arbitrary half-infinite holomorphic curve as in 4.4.5 and let $v = f_t(u) \in H_{bc}^2(\mathbb{R}^\pm \times [0, 1], \mathbb{R}^{2n})$ be also as in 4.4.5 . If*

$$\|v(0)\|_{3/2} < \epsilon \quad (4.58)$$

then

$$\|v\|_{W^{2,2}(\mathbb{R}^\pm \times [0,1])} \leq c \|v(0)\|_{3/2}. \quad (4.59)$$

Here $\|v\|_{3/2}$ denotes the norm of v in the interpolation space $H_{bc}^{3/2}$.

Proof of Theorem 4.4.8. We do the proof in the case of positive half-infinite strips. The case of negative strips is analogous. We first prove in Steps 1 and 2 that $W^{1,2}$ norm of v is bounded above by constant times H^1 norm at the boundary. This follows by combining monotonicity results, exponential decay and the fact that the curve is contained in a local coordinate chart.

Step 1. There exist positive constants c_1 and ϵ_1 with the following significance. If $\|v(0)\|_{H^1} < \epsilon_1$ then

$$\|v\|_{L^\infty(\mathbb{R}^+ \times [0,1])} < c_1 \|v(0)\|_{H^1}.$$

The claim follows from the following facts:

- i) As $v = f_t(u)$ we have that the L^∞ norm of a perturbed holomorphic curve v , $\|v\|_{L^\infty(\mathbb{R}^+ \times [0,1])}$, and $\sup_{s,t} d(u(s,t), p)$ are equivalent.
- ii) In Theorems 2.1.3 and 2.1.5 we have proved that the energy of u , $E(u)$ and $\sup_{t \in [0,1]} d(u(0,t), p)$ control the distance $\sup_{s,t} d(u(s,t), p)$. In other words there exist \hbar and δ such that the following holds: If $E(u) < \hbar$ and $\sup_t d(u(0,t), p) < \delta$ then

$$\sup_{s,t} d(u(s,t), p) < c \cdot \sup_t d(u(0,t), p)$$

iii) Both $E(u)$ and $\sup_t d(u(0, t), p)$ are controlled by $\|v(0)\|_{H^1}$. As u is contained in a local Darboux chart we have that

$$\sqrt{E(u)} \leq c\|v(0)\|_{H^1([0,1])}. \quad (4.60)$$

and also $\sup_t d(u(0, t), p) \leq c\|v(0)\|_{L^\infty} \leq c\|v(0)\|_{H^1}$.

Step 2. There exist ϵ_2 and c_2 such that every v as in the statement of the theorem with the property $\|v(0)\|_{H^1} < \epsilon_2$ satisfies

$$\|v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} \leq c_2\|v(0)\|_{H^1}.$$

As u decays exponentially it follows from Proposition 2.3.1 and Corollary 2.3.2 that $d(u(s, t), p) \leq c\sqrt{E(u)}e^{-\mu s}$ for all $s \geq 1$. Thus it follows combining the facts from step 1, i) and iii) that

$$\|v\|_{L^2([1,+\infty) \times [0,1])} \leq c\|v(0)\|_{H^1}.$$

On the compact piece $[0, 1] \times [0, 1]$ we can estimate the L^2 norm of v by its L^∞ norm, hence the claim of step 2 follows for the L^2 norm of v using the result of step 1). The L^2 norm of $\partial_s u$ and $\partial_s v$ are equivalent, thus using the fact iii) from Step 1 we have

$$\|\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])} \leq c\sqrt{E(u)} \leq c\|v(0)\|_{H^1}.$$

As $\partial_s v + J_t(v)(\partial_t v - X_t(v)) = 0$ and $X_t(0) = 0$ we have

$$\|\partial_t v\|_{L^2(\mathbb{R}^+ \times [0,1])} \leq c\left(\|v\|_{L^2(\mathbb{R}^+ \times [0,1])} + \|\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])}(1 + \|v\|_{L^\infty})\right).$$

Thus $\|v(0)\|_{H^1}$ controls also L^2 norm of $\partial_t v$.

In order to prove theorem 4.4.8 we still need to estimate $\|\partial_s v\|_{W^{1,2}}$ and $\|\partial_t v\|_{W^{1,2}}$. Notice that $\partial_s v \in \text{Ker}(D_v)$, where D_v is the linearized operator as in (4.57).

Step 3. Let $H_{bc}^1(\mathbb{R}^+ \times [0, 1])$ be defined as in (3.9). Let D_0 be the linearization at 0, as in 4.4.6 and let $1 < p < 2$. There exists a constant $c_0 > 0$ such that every $\xi \in H_{bc}^1(\mathbb{R}^+ \times [0, 1], \mathbb{R}^{2n}) \cap W^{1,p}(\mathbb{R}^+ \times [0, 1])$ satisfies the following

$$\|\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0,1])} \leq c_0\left(\|D_0 \xi\|_{L^p} + \|\xi(0)\|_{1/2}\right). \quad (4.61)$$

Define the space $W_{bc}^{1,p}(\mathbb{R} \times [0, 1])$ as follows

$$W_{bc}^{1,p}(\mathbb{R} \times [0, 1]) = \left\{ \eta \in W^{1,p}(\mathbb{R} \times [0, 1], \mathbb{R}^{2n}) \mid \eta(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \right\}$$

Then the following inequality follows

$$\|\eta\|_{W^{1,p}(\mathbb{R} \times [0, 1])} \leq C \|D_0 \eta\|_{L^p(\mathbb{R} \times [0, 1])}. \quad (4.62)$$

For the proof of (4.62) have a look at the L^p estimates in section 3.4.2. Using the inequality (4.62), we are able to prove Step 3. Let $\xi_0 = \xi(0, \cdot) \in [W_{bc}^{1,2}, L^2]_{1/2}$ and let $\eta_0(s, t) \in W^{1,2}(\mathbb{R}^+ \times [0, 1])$ be the extension of $\xi_0(t)$, we can suppose w.l.o.g that η_0 has compact support, thus $\eta_0 \in W^{1,p}(\mathbb{R}^+ \times [0, 1])$ and the following inequality holds

$$\|\eta_0\|_{W^{1,p}(\mathbb{R}^+ \times [0, 1])} \leq c \|\eta_0\|_{W^{1,2}(\mathbb{R}^+ \times [0, 1])} \leq c \|\xi_0\|_{1/2}. \quad (4.63)$$

Let $\zeta_0(s, t) = \xi(s, t) - \eta_0(s, t)$. Then $\zeta_0 \in W^{1,p}(\mathbb{R}^+ \times [0, 1])$ and $\zeta_0(0, \cdot) = 0$. Extend ζ_0 by 0 to the whole of $\mathbb{R} \times [0, 1]$, i.e. define $\zeta(s, t)$ by

$$\zeta(s, t) = \begin{cases} \zeta_0(s, t), & s \geq 0 \\ 0, & s \leq 0 \end{cases}$$

From the inequalities (4.62) and (4.63) we obtain

$$\begin{aligned} \|\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0, 1])} &\leq \|\eta_0\|_{W^{1,p}(\mathbb{R}^+ \times [0, 1])} + \|\zeta\|_{W^{1,p}(\mathbb{R}^+ \times [0, 1])} \\ &\leq \|\eta_0\|_{1,p} + C \|D_0 \zeta\|_{L^p} \\ &\leq \|\eta_0\|_{1,p} + C \|D_0 \xi\|_{L^p} + C \|D_0 \eta_0\|_{L^p} \\ &\leq c_0 \left(\|D_0 \xi\|_{L^p(\mathbb{R}^+ \times [0, 1])} + \|\xi(0)\|_{1/2} \right) \end{aligned}$$

Here the last inequality follows as $\|D_0 \eta_0\|_{L^p} \leq c \|\eta_0\|_{1,p}$ and using the inequality (4.63).

Step 4. Let D_v be the linearized operator as in (4.57). There exists $\delta > 0$ and $c_1 > 0$ such that the following holds. Assume that $\|v(0)\|_{H^1([0, 1])} \leq \delta$, then

$$\|\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0, 1])} \leq c_1 (\|D_v \xi\|_{L^p} + \|\xi(0)\|_{1/2}) \quad (4.64)$$

holds for all $\xi \in W_{bc}^{1,2}(\mathbb{R}^+ \times [0, 1]) \cap W^{1,p}(\mathbb{R}^+ \times [0, 1])$.

We will use the inequality (4.61) proved in Step 2 and we will prove that the norm of the difference $\|(D_v - D_0)\xi\|_{L^p}$ is small provided that $\|v(0)\|_{H^1([0, 1])}$

is sufficiently small. Let

$$\begin{aligned} \|(D_v - D_0)\xi\|_{L^p} &\leq \overbrace{\|(J_t(v) - J_t(0))\partial_t \xi\|_{L^p}}^I \\ &\quad + \overbrace{\|(J_t(v)dX_t(v) - J_t(0)dX_t(0))\xi\|_{L^p}}^{II} \\ &\quad + \overbrace{\|(dJ_t(v)\xi)J_t(v)\partial_s v\|_{L^p}}^{III}. \end{aligned}$$

Obviously $I, II \leq c\|v\|_{L^\infty}\|\xi\|_{W^{1,p}}$ and we have proved in Step 1 that $\|v(0)\|_{H^1}$ controls the L^∞ norm of v . Thus

$$I, II \leq c\|v(0)\|_{H^1}\|\xi\|_{W^{1,p}}.$$

We estimate the III term in the following way. Remember that $\|\partial_s v\|_{L^\infty}$ decays exponentially for $s \geq 1$, from Proposition 2.3.1 we have

$$\|\partial_s v\|_{L^\infty([s,\infty) \times [0,1])} \leq c\sqrt{E(u)}e^{-\mu s} \leq c\|v(0)\|_{H^1([0,1])}e^{-\mu s}, \quad s \geq 1 \quad (4.65)$$

Let $\beta(s)$ be a smooth cut-off function with

$$\beta(s) = \begin{cases} 1, & s \leq 1 \\ 0, & s \geq 2 \end{cases}$$

then

$$\begin{aligned} III &\leq c(1 + \|v\|_{L^\infty})\|(dJ_t(v)\xi)\partial_s v\|_{L^p(\mathbb{R}^+ \times [0,1])} \\ &\leq c\|(dJ_t(v)\xi)(\beta\partial_s v + (1-\beta)\partial_s v)\|_{L^p(\mathbb{R}^+ \times [0,1])} \\ &\leq c(\|\beta(dJ_t(v)\xi)\partial_s v\|_{L^p} + \|(dJ_t(v)\xi)(1-\beta)\partial_s v\|_{L^p}) \\ &\leq c(\|(dJ_t(v)\xi)\partial_s v\|_{L^p([0,2] \times [0,1])} + c'(1 + \|v\|_{L^\infty})\|v(0)\|_{H^1([0,1])}\|\xi\|_{L^p}) \\ &\leq c(1 + \|v\|_{L^\infty}) (\|\xi\|_{L^q([0,2] \times [0,1])}\|\partial_s v\|_{L^2([0,2] \times [0,1])} + \|v(0)\|_{H^1([0,1])}\|\xi\|_{L^p}) \\ &\leq c(1 + \|v\|_{L^\infty})\|v(0)\|_{H^1}\|\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0,1])} \end{aligned}$$

Here the second inequality follows from Step 1 and the assumption $\|v(0)\|_{H^1} < \delta$. In the penultimate inequality we have $q = \frac{2p}{2-p}$ and the inequality follows from Hölder inequality whereas the last inequality is a corollary of the Sobolev embedding $W^{1,p}([0,2] \times [0,1]) \hookrightarrow L^q([0,2] \times [0,1])$ and Step 2. Thus for sufficiently small δ and $\|v(0)\|_{H^1} < \delta$ we have

$$\|(D_v - D_0)\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0,1])} \leq \frac{1}{2c_0}\|\xi\|_{W^{1,p}}, \quad (4.66)$$

where c_0 is the constant in (4.61). Substituting (4.66) in (4.61) we obtain

$$\begin{aligned} \|\xi\|_{W^{1,p}(\mathbb{R}^+ \times [0,1])} &\leq c_0 (\|D_0\xi\|_{L^p(\mathbb{R}^+ \times [0,1])} + \|\xi(0)\|_{1/2}) \\ &\leq c_0 (\|D_v\xi\|_{L^p} + \|(D_v - D_0)\xi\|_{L^p} + \|\xi(0)\|_{1/2}) \\ &\leq 2c_0 (\|D_v\xi\|_{L^p} + \|\xi(0)\|_{1/2}). \end{aligned} \quad (4.67)$$

Thus, we have proved Step 4.

Step 5. There exist $\epsilon_0 > 0$ and $c_0 > 0$ such that the following holds. If $\|v(0)\|_{3/2} < \epsilon_0$ then

$$\begin{aligned} \|\partial_s v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} &< c_0 \|v(0)\|_{3/2} \\ \|\partial_t v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} &< c_0 \|v(0)\|_{3/2} \end{aligned} \quad (4.68)$$

Proof. In Lemma 3.1.6 we have proved that the following holds for all $\xi \in W_{bc}^{1,2}(\mathbb{R}^+ \times [0,1])$

$$\|\xi\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} \leq c (\|D_0\xi\|_{L^2(\mathbb{R}^+ \times [0,1])} + \|\xi(0)\|_{1/2}). \quad (4.69)$$

where D_0 is the linearization as in 4.4.6. We want to estimate $\|\partial_s v\|_{1,2}$. Notice that $\partial_s v$ is an element of the kernel of the operator D_v . In order to estimate $\|\partial_s v\|_{1,2}$ we need to estimate the difference $\|(D_v - D_0)\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])}$. We have

$$\begin{aligned} \|(D_v - D_0)\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])} &\leq \overbrace{\|(J_t(v) - J_t(0))\partial_s \partial_t v\|_{L^2(\mathbb{R}^+ \times [0,1])}}^I \\ &\quad + \overbrace{\|(J_t(v)dX_t(v) - J_t(0)dX_t(0))\partial_s v\|_{L^2}}^{II} \\ &\quad + \overbrace{\|(dJ_t(v)\partial_s v)J_t(v)\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])}}^{III} \end{aligned} \quad (4.70)$$

Obviously it follows from Steps 1 and 2 that

$$\begin{aligned} I &\leq c\|v\|_{L^\infty} \|\partial_s v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} \leq c\|v(0)\|_{H^1} \|\partial_s v\|_{W^{1,2}} \\ II &\leq c\|v\|_{L^\infty} \|\partial_s v\|_{L^2(\mathbb{R}^+ \times [0,1])} \leq c\|v(0)\|_{H^1}^2 \end{aligned} \quad (4.71)$$

whereas

$$III \leq c(1 + \|v\|_{L^\infty})^2 \|\partial_s v\|_{L^4(\mathbb{R}^+ \times [0,1])}^2.$$

Let $\beta(s)$ be a smooth cut-off function as in Step 4, then

$$\|(1 - \beta)\partial_s v\|_{L^4(\mathbb{R}^+ \times [0,1])}^2 \leq c\|v(0)\|_{H^1([0,1])}^2,$$

as $|\partial_s v(s, t)| \leq c \|v(0)\|_{H^1([0,1])} e^{-\mu s}$ for $s \geq 1$. On the other hand, from the Sobolev embedding, $W^{1,4/3}([0, 2] \times [0, 1]) \subset L^4([0, 2] \times [0, 1])$, we have

$$\|\beta \partial_s v\|_{L^4} \leq c \|\beta \partial_s v\|_{W^{1,4/3}},$$

for some positive constant c . Suppose that ϵ_0 is chosen such that $\|v(0)\|_{H^1} < \delta$, where δ is the constant in the claim of Step 4. Substituting $\beta \partial_s v$ in the inequality (4.64) with $p = 4/3$ we obtain

$$\begin{aligned} \|\beta \partial_s v\|_{W^{1,4/3}(\mathbb{R}^+ \times [0,1])} &\leq c \left(\|D_v(\beta \partial_s v)\|_{L^{4/3}} + \|\partial_s v(0)\|_{1/2} \right) \\ &\leq c \left(\|\beta D_v(\partial_s v)\|_{L^{4/3}} + \|\dot{\beta} \partial_s v\|_{L^{4/3}} + \|\partial_s v(0)\|_{1/2} \right) \\ &\leq c \left(\|\partial_s v\|_{L^2([0,2] \times [0,1])} + \|\partial_s v(0)\|_{1/2} \right) \end{aligned}$$

Now we can estimate the *III* term with

$$\begin{aligned} III &\leq c(1 + \|v\|_{L^\infty})^2 \cdot \|(\beta + (1 - \beta))\partial_s v\|_{L^4}^2 \\ &\leq c(1 + \|v\|_{L^\infty})^2 \left(\|\beta \partial_s v\|_{L^4}^2 + \|(1 - \beta)\partial_s v\|_{L^4}^2 \right) \\ &\leq c(1 + \|v\|_{L^\infty})^2 \left(\|\partial_s v\|_{L^2([0,2] \times [0,1])}^2 + \|\partial_s v(0)\|_{1/2}^2 + \|v(0)\|_{H^1([0,1])}^2 \right) \\ &\leq c(1 + \|v\|_{L^\infty})^2 \left(\|v(0)\|_{H^1}^2 + \|\partial_s v(0)\|_{1/2}^2 \right). \end{aligned}$$

Finally we estimate $\|\partial_s v(0)\|_{1/2}$. As v is a solution of the equation (4.53) we have:

$$\begin{aligned} \|\partial_s v(0)\|_{1/2} &= \|J_t(v(0))(\partial_t v(0) - X_t(v(0)))\|_{1/2} \\ &\leq \overbrace{\|J_t(v(0))\partial_t v(0)\|_{1/2}}^a + \overbrace{\|J_t(v(0))X_t(v(0))\|_{1/2}}^b \end{aligned} \quad (4.72)$$

To estimate the terms a and b we shall use the fact that

$$\|fg\|_{1/2} \leq c \|f\|_{H^1} \cdot \|g\|_{1/2}$$

for $f \in H^1([0, 1], \mathbb{R})$ and $g \in [W_{bc}^{1,2}, L^2]_{1/2}$. This holds as multiplication by f is a continuous linear map on both $W_{bc}^{1,2}$ and L^2 , hence it is also continuous on $[W_{bc}^{1,2}, L^2]_{1/2}$. Thus, as $X_t(0) = 0$ it follows that

$$\begin{aligned} a &\leq c(1 + \|v(0)\|_{H^1}) \|v(0)\|_{3/2} \\ b &\leq c(1 + \|v(0)\|_{H^1}) \|v(0)\|_{H^1} \end{aligned}$$

As we can w.l.o.g. assume that $\|v(0)\|_{H^1}$ is small we obtain from the previous inequality and (4.72)

$$\|\partial_s v(0)\|_{1/2} \leq c\|v(0)\|_{3/2}. \quad (4.73)$$

Substituting $\partial_s v$ in the inequality (4.69) and using the estimates for I, II and III term we obtain

$$\begin{aligned} \|\partial_s v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} &\leq c \left(\|D_0 \partial_s v\|_{L^2} + \|\partial_s v(0)\|_{1/2} \right) \\ &\leq c \left(\|D_v \partial_s v\|_{L^2} + \|(D_v - D_0) \partial_s v\|_{L^2} + \|v(0)\|_{3/2} \right) \\ &\leq c \left(I + II + III + \|v(0)\|_{3/2} \right) \\ &\leq c \left((1 + \|v\|_{L^\infty})^2 \|v(0)\|_{H^1} + \|v(0)\|_{3/2} \right). \end{aligned} \quad (4.74)$$

Here the second inequality follows from (4.73). From the inequality (4.74) and Step 1 follows the inequality (4.68) for $\partial_s v$. As v is the solution of (4.53), we have

$$\|\partial_t v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} \leq \|X_t(v)\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} + \|J_t(v) \partial_s v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])}.$$

As $X_t(0) = 0$ and X_t is smooth it follows that $\|X_t(v)\|_{1,2} \leq c(1 + \|v\|_{L^\infty})\|v\|_{1,2}$. Also,

$$\begin{aligned} \|J_t(v) \partial_s v\|_{W^{1,2}} &\leq \|J_t(v) \partial_s v\|_{L^2} + \|\partial_t J_t(v) \partial_s v\|_{L^2} \\ &\quad + \|(dJ_t(v) \partial_t v) \partial_s v\|_{L^2} + \|(dJ_t(v) \partial_s v) \partial_s v\|_{L^2} \\ &\quad + \|J_t(v) \partial_s^2 v\|_{L^2} + \|J_t(v) \partial_s \partial_t v\|_{L^2} \\ &\leq c(1 + \|v\|_{L^\infty}) \|\partial_s v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} + \|(dJ_t(v) \partial_t v) \partial_s v\|_{L^2} \\ &\quad + \|(dJ_t(v) \partial_s v) \partial_s v\|_{L^2} \end{aligned}$$

Notice that

$$\|(dJ_t(v) \partial_t v) \partial_s v\|_{L^2} \leq \|(dJ_t(v) X_t(v)) \partial_s v\|_{L^2} + \|(dJ_t(v) J_t(v) \partial_s v) \partial_s v\|_{L^2}.$$

Using the same type of estimates as for the III term we obtain the analog of (4.74) for $\partial_t v$, i.e.

$$\|\partial_t v\|_{W^{1,2}(\mathbb{R}^+ \times [0,1])} \leq c \left((1 + \|v\|_{L^\infty})^2 \|v(0)\|_{H^1} + \|v(0)\|_{3/2} \right). \quad (4.75)$$

□

The steps 1-5 prove the theorem 4.4.8. □

Remark 4.4.9. The analogous statement as in Theorem 4.4.8 holds for finite strips. Namely, there exist $\epsilon > 0, c > 0$ such that the following holds for every T . If $v = f_t(u) \in H_{bc}^2([-T, T] \times [0, 1], \mathbb{R}^{2n})$, as in local setup 4.4.5 satisfies

$$\|v(-T)\|_{3/2} + \|v(T)\|_{3/2} < \epsilon,$$

then

$$\|v\|_{W^{2,2}([-T,T] \times [0,1])} < c(\|v(-T)\|_{3/2} + \|v(T)\|_{3/2})$$

The proof is analog to the proof of proposition 4.4.8 and we shall not repeat it.

4.5 Convergence theorem

In this section we prove Theorem 4.1.8.

4.5.1 (Hardy submanifolds). In Definition 4.4.4 we have introduced manifolds $\mathcal{M}_\epsilon^\infty$ and \mathcal{M}_ϵ^T and we have proved in 4.4.7 that these manifolds are embedded submanifolds of the Hilbert manifolds of paths $\mathcal{P}^{3/2} \times \mathcal{P}^{3/2}$. Denote with $\mathcal{W}_\epsilon^\infty$ and \mathcal{W}_ϵ^T the images of $\mathcal{M}_\epsilon^\infty$ and \mathcal{M}_ϵ^T via the maps i^∞ and i^T .

$$\begin{aligned} i^T : \mathcal{M}_\epsilon^T &\rightarrow \mathcal{W}_\epsilon^T \subset \mathcal{P}^{3/2} \times \mathcal{P}^{3/2}, \quad i^T(u) = (u(-T, \cdot), u(T, \cdot)) \\ i^\infty : \mathcal{M}_\epsilon^\infty &\rightarrow \mathcal{W}_\epsilon^\infty \subset \mathcal{P}^{3/2} \times \mathcal{P}^{3/2}, \quad (u^-, u^+) \mapsto (u^-(0, \cdot), u^+(0, \cdot)) \end{aligned} \quad (4.76)$$

These are **Hardy** submanifolds of the path space and we can think of them as of those paths that extend holomorphically to the corresponding strips.

$$\begin{aligned} \mathcal{W}_\epsilon^\infty &= \left\{ (\gamma^-, \gamma^+) \in \mathcal{U}_\epsilon(p) \times \mathcal{U}_\epsilon(p) \mid \exists (u^+, u^-) \in \mathcal{M}_\epsilon^\infty, \quad u^\pm(0, \cdot) = \gamma^\mp(\cdot) \right\} \\ &= \left\{ (\gamma^-, \gamma^+) \in \mathcal{U}_\epsilon(p) \times \mathcal{U}_\epsilon(p) \mid \exists u^\pm \in H_{loc}^2(\mathbb{R}^\pm \times [0, 1], N) \right. \\ &\quad \left. u(s, i) \in L_i, \quad i = 0, 1, \quad \forall s \right. \\ &\quad \left. \bar{\partial}_{J_t} u^\pm = 0, \quad E(u^\pm) < \hbar \right. \\ &\quad \left. u^\pm(0, \cdot) = \gamma^\mp(\cdot) \right\} \end{aligned}$$

and similarly

$$\mathcal{W}_\epsilon^T = \left\{ (\gamma^-, \gamma^+) \in \mathcal{U}_\epsilon(p) \times \mathcal{U}_\epsilon(p) \mid \exists u \in \mathcal{M}_\epsilon^T, \quad u(\pm T, \cdot) = \gamma^\pm(\cdot) \right\}.$$

We prove that \mathcal{W}_ϵ^T converge to $\mathcal{W}_\epsilon^\infty$ in C^1 topology.

4.5.2 (Notion of convergence of Hilbert manifolds). We first explain the notion of convergence of certain Hilbert manifolds. Let W^T and W^∞ be some Hilbert manifolds. There are different ways one can think about C^1 convergence $W^T \rightarrow W^\infty$. One way is to think of W^T as of the sections of the normal bundle of W^∞ and to prove that these sections converge to the zero section. Another way is to think of representation of W^T and W^∞ in a local chart and to prove that locally W^T converges to W^∞ . We formulate this more precisely in the next definition.

Definition 4.5.3. *Let P be a Hilbert manifold modeled on Hilbert space H and let W^∞ and W^T be its submanifolds. We say that $W^T \rightarrow W^\infty$, $T \rightarrow +\infty$ if for all $x_0 \in W^\infty$ there exist the following:*

- 1) *A splitting $H = H_0 \oplus H_1$, where H_0 and H_1 are closed subspaces of H .*
- 2) *A local coordinate chart $\phi : U \rightarrow H$, where $U \subset P$ is an open neighborhood of x_0 , such that $\phi(x_0) = 0$ and*

$$\phi(U) = \{\xi_0 + \xi_1 : \xi_0 \in U_0, \xi_1 \in U_1\},$$

where $U_0 \subset H_0$ and $U_1 \subset H_1$ are open neighborhoods of 0.

- 3) *A smooth map $f^\infty : U_0 \rightarrow H^1$ such that*

$$\phi(U \cap W^\infty) = \{\xi + f^\infty(\xi) : \xi \in U_0\}.$$

A family of smooth maps $f^T : U_0 \rightarrow H^1$ such that for all $T \geq T_0$

$$\phi(U \cap W^T) = \{\xi + f^T(\xi) : \xi \in U_0\}.$$

- 4) *The limits $\lim_{T \rightarrow \infty} \|f^T - f^\infty\|_{C^1} = 0$*

In the proof of the Theorem 4.1.8 we shall often use the inverse function theorem, we state it in the form that we shall use in the proof.

Lemma 4.5.4 (Inverse function theorem). *Let X and Y be Banach spaces, $B_r(x_0) \subset X$ a ball of radius r and $f : B_r(x_0) \rightarrow Y$ a continuously differentiable function that satisfies the following:*

- 1) *$df(x_0)$ is bijective and the operator norm $\|df(x_0)^{-1}\| \leq c$.*
- 2) *$\|df(x) - df(x_0)\| \leq \frac{1}{2c}$, for all $x \in B_r(x_0)$.*

Then $f : B_r(x_0) \xrightarrow{dff} f(B_r(x_0)) \subset Y$ and $B_{r/2c}(f(x_0)) \subset f(B_r(x_0))$.

Proof of Theorem 4.1.8. Let $\mathcal{W}_\epsilon^\infty$ and \mathcal{W}_ϵ^T be as in 4.5.1. Remember that they are embedded submanifolds of the Hilbert manifold of paths $\mathcal{P}^{3/2} \times \mathcal{P}^{3/2}$. Let

$$\begin{aligned}\mathcal{W}_\epsilon^\infty &= (\Phi_p \times \Phi_p)(\mathcal{W}_\epsilon^\infty) \\ \mathcal{W}_\epsilon^T &= (\Phi_p \times \Phi_p)(\mathcal{W}_\epsilon^T)\end{aligned}\tag{4.77}$$

where $\Phi_p : \mathcal{U}_p \rightarrow \mathcal{W}_p \subset H_{bc}^{3/2} = E$ is a local chart as in 4.4.2. We devide the proof in two parts. In the first part, i.e. in 4.5.5, we shall construct maps

$$f^\infty, f^T : B_{\rho_0}(0) \rightarrow E,$$

which are diffeomorphisms onto their images and $B_{\rho_0}(0)$, is an open ball of radius ρ_0 centered at 0 in E . The set $\mathcal{W}_\epsilon^\infty$ is an open subset of $\text{graph}(f^\infty)$ and also \mathcal{W}_ϵ^T is an open subset of $\text{graph}(f^T)$. In the second part i.e. in 4.5.6 we prove the convergence

$$f^T \xrightarrow{C^1} f^\infty, \quad T \rightarrow \infty$$

on some smaller neighborhood $B_\rho(0) \subset B_{\rho_0}(0)$. And finally the convergence of maps $f^T \rightarrow f^\infty$ will imply the convergence of submanifolds $\mathcal{W}_\epsilon^T \rightarrow \mathcal{W}_\epsilon^\infty$. \square

4.5.5 (Construction of the maps f^T and f^∞). Let D_0 be the linearization at 0 as in 4.4.6. Then $D_0 = \partial_s + A$ and the operator

$$A : H_{bc}^1([0, 1]) \rightarrow L^2([0, 1]), \quad A\xi = J_t(0)\partial_t\xi - J_t(0)dX_t(0)\xi$$

satisfies (HA) in 3.1.1 as it is self adjoint with respect to the scalar product given by (4.56). Let $E = H_{bc}^{3/2}$, E^\pm be as in Remark 3.1.3, corresponding to the operator A , and let π^\pm be as in (3.7). Abbreviate

$$Z^\pm = \mathbb{R}^\pm \times [0, 1], \quad Z^T = [-T, T] \times [0, 1].$$

Let $H_{bc}^i(Z^\pm)$ and $H_{bc}^i(Z^T)$, $i = 1, 2$ be defined as in (3.9) and (3.11). We define maps \mathcal{F}^∞ and \mathcal{F}^T as follows

$$\begin{aligned}\mathcal{F}^\infty : H_{bc}^2(Z^+) \times H_{bc}^2(Z^-) &\longrightarrow H_{bc}^1(Z^+) \times H_{bc}^1(Z^-) \times E^+ \cap E \times E^- \cap E \\ \mathcal{F}^\infty(u^-, u^+) &= \left(\bar{\partial}_{J_t, X_t} u^-, \bar{\partial}_{J_t, X_t} u^+, \pi^+(u^-(0, \cdot)), \pi^-(u^+(0, \cdot)) \right)\end{aligned}\tag{4.78}$$

$$\begin{aligned}\mathcal{F}^T : H_{bc}^2(Z^T) &\longrightarrow H_{bc}^1(Z^T) \times E^+ \times E^- \\ \mathcal{F}^T(u) &= \left(\bar{\partial}_{J_t, X_t} u, \pi^+(u(-T, \cdot)), \pi^-(u(T, \cdot)) \right).\end{aligned}\tag{4.79}$$

Here $\bar{\partial}_{J_t, X_t}$ is as in (4.53).

Step 1: Let \mathcal{F}^∞ and \mathcal{F}^T be as in (4.78) and (4.79). There exist $c_0 > 0$ such that the maps $d\mathcal{F}^\infty(0)$ and $d\mathcal{F}^T(0)$ are bijective and have uniformly bounded inverses

$$\|d\mathcal{F}^\infty(0)^{-1}\| \leq c_0, \quad \|d\mathcal{F}^T(0)^{-1}\| \leq c_0, \quad (4.80)$$

Proof. Notice that the function $\mathcal{F}^\infty = \mathcal{F}^+ \times \mathcal{F}^-$, where

$$\begin{aligned} \mathcal{F}^\pm &: H_{bc}^2(Z^\pm) \rightarrow H_{bc}^1(Z^\pm) \times E^\pm \cap E \\ \mathcal{F}^\pm(u) &= (\bar{\partial}_{J_t, X_t} u, \pi^\pm(u(0, \cdot))) \end{aligned}$$

The linearizations of \mathcal{F}^∞ and \mathcal{F}^T at 0 are given by:

$$\begin{aligned} d\mathcal{F}^\infty(0)(\hat{u}^-, \hat{u}^+) &= (D_0 \hat{u}^-, D_0 \hat{u}^+, \pi^+(\hat{u}^-(0, \cdot)), \pi^-(\hat{u}^+(0, \cdot))) \\ d\mathcal{F}^T(0)(\hat{u}) &= (D_0 \hat{u}, \pi^+(\hat{u}(-T, \cdot)), \pi^-(\hat{u}(T, \cdot))). \end{aligned}$$

The fact that the maps $d\mathcal{F}^\infty(0)$ and $d\mathcal{F}^T(0)$ are bijective and the inequality (4.80) follow directly from Theorem 3.1.6 \square

Step 2: [Quadratic estimates] Let c_0 be the constant as in Step 1. There exist $r_0 > 0$ such that for all $(u^-, u^+) \in H_{bc}^2(Z^+) \times H_{bc}^2(Z^-)$ and all $u \in H_{bc}^2(Z^T)$, which satisfy

$$\|u^-\|_{2,2} + \|u^+\|_{2,2} < r_0, \quad \|u\|_{2,2} < r_0 \quad (4.81)$$

the following holds.

$$\|d\mathcal{F}^\infty(u^-, u^+) - d\mathcal{F}^\infty(0, 0)\| \leq \frac{1}{2c_0}, \quad \|d\mathcal{F}^T(u) - d\mathcal{F}^T(0)\| \leq \frac{1}{2c_0}, \quad (4.82)$$

Proof. Notice that

$$\begin{aligned} \|d\mathcal{F}^T(u)(\hat{u}) - d\mathcal{F}^T(0)(\hat{u})\| &= \|(D_u - D_0)\hat{u}\|_{1,2} \\ \|(d\mathcal{F}^\infty(u^+, u^-) - d\mathcal{F}^\infty(0))(\hat{u}^+, \hat{u}^-)\| &= \|(D_{u^+} - D_0)(\hat{u}^+)\|_{1,2} \\ &\quad + \|(D_{u^-} - D_0)(\hat{u}^-)\|_{1,2}, \end{aligned}$$

Hence, we need to estimate the difference $\|(D_u - D_0)\hat{u}\|_{1,2}$, for $u \in H_{bc}^2(I \times [0, 1])$, where D_u is the linearized operator as in (4.57), i.e.

$$D_u \hat{u} = \partial_s \hat{u} + J_t(u)(\partial_t \hat{u} - dX_t(u)\hat{u}) + (dJ_t(u)\hat{u})(\partial_t u - X_t(u)).$$

We prove that for all $u, \hat{u} \in H_{bc}^2(\mathbb{R}^+ \times [0, 1])$ with $\|u\|_{W^{2,2}(\mathbb{R}^+ \times [0,1])} \leq 1$ we have

$$\|(D_u - D_0)\hat{u}\|_{1,2} \leq c\|u\|_{2,2}\|\hat{u}\|_{2,2} \quad (4.83)$$

for some positive constant c . The same inequality holds for finite strips and the proof is analogous.

Let $\beta_i : \mathbb{R} \rightarrow [0, 1]$, $i \geq 0$ be smooth cut-off functions, with the properties

- a) $\text{supp}(\beta_i) \subset [i, i+2]$, $i \geq 1$, and $\beta_0(s) = 0$ for $s \geq 2$
- b) $\sum_{i=0}^{+\infty} \beta_i(s) = 1$ and $\|\dot{\beta}_i(s)\|_{C^0} \leq 1$, $i \geq 0$.

Let $\hat{u} = \sum_{i=0}^{\infty} \beta_i \hat{u} = \sum_{i=0}^{+\infty} \hat{u}_i$. We have

$$\begin{aligned} \|(D_u - D_0)(\sum_i \hat{u}_i)\|_{W^{1,2}([0,+\infty) \times [0,1])} &= \|\sum_i (D_u - D_0)\hat{u}_i\|_{W^{1,2}([0,+\infty) \times [0,1])} \\ &\leq \sum_i \|(D_u - D_0)\hat{u}_i\|_{W^{1,2}([i,i+2] \times [0,1])} \end{aligned}$$

Suppose that the inequality

$$\|(D_u - D_0)\hat{u}_i\|_{W^{1,2}([i,i+2] \times [0,1])} \leq c\|u\|_{W^{2,2}([i,i+2] \times [0,1])} \cdot \|\hat{u}_i\|_{W^{2,2}([i,i+2] \times [0,1])} \quad (4.84)$$

holds for some positive constant c and for all $i \geq 0$. With this assumption we have

$$\begin{aligned} \|(D_u - D_0)\hat{u}\|_{2,2} &\leq \sum_i \|(D_u - D_0)\hat{u}_i\|_{W^{1,2}([i,i+2] \times [0,1])} \\ &\leq c \sum_i \|u\|_{W^{2,2}([i,i+2] \times [0,1])} \cdot \|\hat{u}_i\|_{W^{2,2}([i,i+2] \times [0,1])} \\ &\leq c \sqrt{\sum_{i \geq 0} \|u\|_{W^{2,2}([i,i+2] \times [0,1])}^2} \sqrt{\sum_{i \geq 0} \|\hat{u}_i\|_{W^{2,2}([i,i+2] \times [0,1])}^2} \\ &\leq c' \|u\|_{W^{2,2}([0,+\infty) \times [0,1])} \|\hat{u}\|_{W^{2,2}([0,+\infty) \times [0,1])} \end{aligned} \quad (4.85)$$

It remains to prove the inequality (4.84) under the assumption $\|u\|_{W^{2,2}(\mathbb{R}^+ \times [0,1])} \leq 1$. Let $\Omega = [i, i+2] \times [0, 1]$, then

$$\begin{aligned}
 \|(D_u - D_0)\hat{u}\|_{W^{1,2}(\Omega)} &\leq \overbrace{\|(J_t(u) - J_t(0))\partial_t \hat{u}\|_{1,2}}^I \\
 &\quad + \overbrace{\|(J_t(u)dX_t(u) - J_t(0)dX_t(0))\hat{u}\|_{1,2}}^{II} \\
 &\quad + \overbrace{\|(dJ_t(u)\hat{u})\partial_t u\|_{1,2}}^{III} + \overbrace{\|(dJ_t(u)\hat{u})X_t(u)\|_{1,2}}^{IV} \quad (4.86)
 \end{aligned}$$

We shall show how to estimate terms I and III , and analogously one can estimate the II and IV term of (4.86).

$$\begin{aligned}
 I &\leq \overbrace{\|(J_t(u) - J_t(0))\partial_t \hat{u}\|_{L^2}}^A + \overbrace{\|(\partial_t J_t(u) - \partial_t J_t(0))\hat{u}\|_{L^2}}^B + \overbrace{\|(dJ_t(u)\partial_t u)\partial_t \hat{u}\|_{L^2}}^C \\
 &\quad + \overbrace{\|(J_t(u) - J_t(0))\partial_t^2 \hat{u}\|_{L^2}}^D + \overbrace{\|(dJ_t(u)\partial_s u)\partial_t \hat{u}\|_{L^2}}^E + \overbrace{\|(J_t(u) - J_t(0))\partial_t \partial_s \hat{u}\|_{L^2}}^F \quad (4.87)
 \end{aligned}$$

The terms A, B, D and F can be estimated by

$$c\|u\|_{L^\infty(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)} \leq c\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}.$$

whereas the terms C and E can be estimated as follows

$$\begin{aligned}
 C &\stackrel{cs}{\leq} c(1 + \|u\|_{L^\infty})\|\partial_t u\|_{L^4}\|\partial_t \hat{u}\|_{L^4} \leq c'\|\partial_t u\|_{W^{1,2}(\Omega)}\|\partial_t \hat{u}\|_{W^{1,2}(\Omega)} \\
 &\leq c'\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}.
 \end{aligned}$$

and the second inequality follows from the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)$. Thus we have proved the desired inequality for the term I

$$I \leq c\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}.$$

We prove an analog inequality for III .

$$\begin{aligned}
 \|(dJ_t(u)\hat{u})\partial_t u\|_{W^{1,2}(\Omega)} &\leq \overbrace{\|(dJ_t(u)\hat{u})\partial_t u\|_{L^2}}^a + \overbrace{\|(\partial_t dJ_t(u)\hat{u})\partial_t u\|_{L^2}}^b \\
 &\quad + \overbrace{\|(d^2 J_t(u)\partial_t u)\hat{u})\partial_t u\|_{L^2}}^d + \overbrace{\|((dJ_t(u)\partial_s u)\hat{u})\partial_t u\|_{L^2}}^e \\
 &\quad + \overbrace{\|(dJ_t(u)\partial_t \hat{u})\partial_t u\|_{L^2}}^f + \overbrace{\|(dJ_t(u)\partial_s \hat{u})\partial_t u\|_{L^2}}^g + \\
 &\quad + \overbrace{\|(dJ_t(u)\hat{u})\partial_t^2 u\|_{L^2}}^h + \overbrace{\|(dJ_t(u)\hat{u})\partial_t \partial_s u\|_{L^2}}^k
 \end{aligned}$$

We have

$$a, b \leq c'(1 + \|u\|_{L^\infty})\|\hat{u}\|_{L^\infty}\|\partial_t u\|_{L^2} \leq c\|\hat{u}\|_{W^{2,2}(\Omega)}\|u\|_{W^{2,2}(\Omega)}.$$

and also

$$\begin{aligned} d &\leq c(1 + \|u\|_{L^\infty})\|\hat{u}\|_{L^\infty}\|\partial_t u\|_{L^4} \leq c\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}. \\ e &\leq c(1 + \|u\|_{L^\infty})\|\hat{u}\|_{L^\infty}\|\partial_s u\|_{L^4}\|\partial_t u\|_{L^4} \leq c\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}. \end{aligned}$$

The terms f and g can be estimated in the same way as the term C , and finally

$$h, k \leq c(1 + \|u\|_{L^\infty})\|\hat{u}\|_{L^\infty(\Omega)}\|u\|_{W^{2,2}(\Omega)} \leq c\|u\|_{W^{2,2}(\Omega)}\|\hat{u}\|_{W^{2,2}(\Omega)}.$$

Thus, we have proved the inequality (4.83). Take $r_0 = \frac{1}{2c_0c}$, where c is the constant from (4.83) and c_0 as in Step 1. For such r_0 the inequality (4.82) is fulfilled. \square

Step 3: Constructions of the maps f^T and f^∞

In Steps 1 and 2 we have proved that the maps \mathcal{F}^∞ and \mathcal{F}^T satisfy properties 1) and 2) of the inverse function theorem, i.e. Lemma 4.5.4. Let $\rho_0 = \frac{r_0}{2c_0}$, where r_0 and c_0 are the constants as in Step 2. For $\xi = (\xi^+, \xi^-) \in B_{\rho_0}(0)$, let

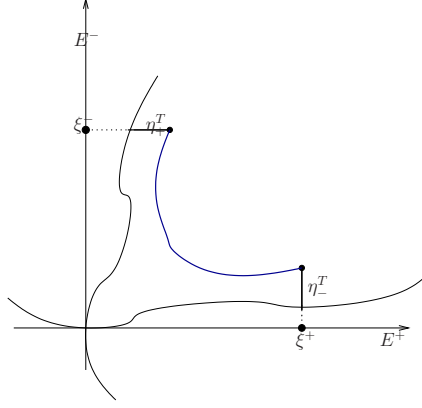
$$u^T = (\mathcal{F}^T)^{-1}(0, \xi), \quad (u^-, u^+) = (\mathcal{F}^\infty)^{-1}(0, 0, \xi^+, \xi^-)$$

We define maps f^∞ and f^T as follows

$$\begin{aligned} f^\infty : B_{\rho_0}(0) &\rightarrow E \\ f^\infty(\xi^+, \xi^-) &= (f^-(\xi^+), f^+(\xi^-)) = (\pi^-(u^-(0, \cdot)), \pi^+(u^+(0, \cdot))) \\ f^T : B_{\rho_0}(0) &\rightarrow E \\ f^T(\xi^+, \xi^-) &= (f_-^T(\xi^+, \xi^-), f_+^T(\xi^+, \xi^-)) = (\pi^-(u^T(-T, \cdot)), \pi^+(u^T(T, \cdot))) \end{aligned} \tag{4.88}$$

The maps f^∞ and f^T are diffeomorphisms onto their images. The sets $\mathcal{W}_{\rho_0}^\infty$ and $\mathcal{W}_{\rho_0}^T$ are open subsets of $\text{graph}(f^\infty)$ and $\text{graph}(f^T)$.

4.5.6 (Convergence $f^T \xrightarrow{C^1} f^\infty$). We have constructed maps f^\pm such that locally the stable and unstable manifolds are graphs of these functions. One can think of the graph of the map f^T for some fixed T as of the darkened line in figure 4.1. This is a bit misleading as $\text{graph}(f^T) \subset E \times E$ and the


 Figure 4.1: Convergence of submanifolds W^T to W^∞

whole picture lies in E . Still the picture gives us good intuition about the convergence phenomenon. We actually have to prove that the difference of the maps f^T and f^∞ , denoted by η_\pm^T in the Figure 4.1 converges to 0.

We have used the inverse function theorem to find the maps f^T and f^∞ and we cannot explicitly say what are these functions. We construct a map that gives their difference $\eta^T = f^T - f^\infty$. Let $B_{\rho_0}(0) \subset E$ be the neighborhood as in Step 3, i.e. such that the maps f^∞ and f^T are defined on $B_{\rho_0}(0)$. Let

$$\begin{aligned} \mathcal{F}^T : B_{\rho_0}(0) \times H_{bc}^2(Z^T) \times E^- \times E^+ &\longrightarrow H^1(Z^T) \times E \times E \\ \mathcal{F}^T(\xi, u, \eta^-, \eta^+) &= \left(\bar{\partial}_{J_t, X_t} u, \xi^+ + f^+(\xi^+) + \eta^- - u(-T, \cdot), \right. \\ &\quad \left. \xi^- + f^-(\xi^-) + \eta^+ - u(T, \cdot) \right), \end{aligned}$$

where f^\pm are as in (4.88). Denote by $\mathcal{F}_\xi^T := \mathcal{F}^T(\xi, \cdot)$. We shall prove in Lemma 4.5.7 that the mapping \mathcal{F}_ξ^T is a local diffeomorphism. Then in Lemma 4.5.8 we construct some possibly smaller neighborhood $B_\rho(0) \subset B_{\rho_0}(0) \subset E$ such that for every $\xi \in B_\rho(0)$ we can find unique $(u^T, \eta_-^T, \eta_+^T, +)$ such that $\mathcal{F}_\xi^T(u^T, \eta_-^T, \eta_+^T) = 0$ and we prove that $\|\eta_-^T\|_{C^1} + \|\eta_+^T\|_{C^1} \rightarrow 0$. Thus we prove that $f^T \rightarrow f^\infty$ as $T \rightarrow +\infty$.

Lemma 4.5.7. *Let ρ_0 be as in Step 3. There exist positive constants C_1, r_1 such that for all $u \in H_{bc}^2(Z^T)$ with*

$$\|u\|_{2,2} < r_1,$$

and all $\xi = (\xi^+, \xi^-) \in B_{\rho_0}(0)$ the following holds

a) The operator $d\mathcal{F}_\xi^T(u, 0, 0)$ is bijective and $\|d\mathcal{F}_\xi^T(u, 0, 0)^{-1}\| \leq C_1$.

b) For all v , $\|v - u\| < r_1$ and for all $\eta^\pm \in E^\pm$ we have

$$\|d\mathcal{F}_\xi^T(u, 0, 0) - d\mathcal{F}_\xi^T(v, \eta^-, \eta^+)\| < \frac{1}{2C_1}$$

Proof of Lemma 4.5.7 :

The derivative of the operator \mathcal{F}_ξ^T at the point $(u, \eta^-, \eta^+) \in H_{bc}^2(Z^T) \times E^- \times E^+$ is given by

$$d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+) = \left(D_u \hat{u}, \hat{\eta}^- - \hat{u}(-T, \cdot), \hat{\eta}^+ - \hat{u}(T, \cdot) \right), \quad (4.89)$$

where the operator D_u is as in (4.57). Notice that the linearization $d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)$ doesn't depend at all on ξ^\pm and η^\pm .

Step A: For all $T > 0$, $\xi \in B_{\rho_0}(0)$ the mapping $d\mathcal{F}_\xi^T(0)$ is bijective. Moreover, there exist a constant $c_1 > 0$ such that

$$\|d\mathcal{F}_\xi^T(0)^{-1}\| \leq c_1 \quad (4.90)$$

Proof. Let D_0 be the linearization of $\bar{\partial}_{J_t, X_t}$ in 0, i.e. D_0 is given as in 4.4.6. From Theorem 3.1.6 we have that for all T

$$\|\hat{u}\|_{2,2} \leq c \left(\|D_0 \hat{u}\|_{1,2} + \|\pi^+(\hat{u}(-T, \cdot))\|_{3/2} + \|\pi^-(\hat{u}(T, \cdot))\|_{3/2} \right), \quad (4.91)$$

and also

$$\begin{aligned} \|\hat{\eta}^-\|_{3/2} &\leq \|\pi^-(\hat{\eta}^- - \hat{u}(-T, \cdot))\|_{3/2} + \|\pi^-(\hat{u}(-T, \cdot))\|_{3/2}, \\ \|\hat{\eta}^+\|_{3/2} &\leq \|\pi^+(\hat{\eta}^+ - \hat{u}(T, \cdot))\|_{3/2} + \|\pi^+(\hat{u}(T, \cdot))\|_{3/2}. \end{aligned} \quad (4.92)$$

Notice also that

$$\begin{aligned} \|\pi^+(\hat{u}(-T, \cdot))\|_{3/2} &= \|\pi^+(\hat{u}(-T, \cdot) - \hat{\eta}^-)\|_{3/2} \\ \|\pi^-(\hat{u}(T, \cdot))\|_{3/2} &= \|\pi^-(\hat{u}(T, \cdot) - \hat{\eta}^+)\|_{3/2} \end{aligned} \quad (4.93)$$

Let

$$L = \|\hat{u}\|_{2,2} + \|\hat{\eta}^-\|_{3/2} + \|\hat{\eta}^+\|_{3/2},$$

then

$$\begin{aligned} L &\leq c \left(\|D_0 \hat{u}\|_{1,2} + \|\hat{\eta}^- - \hat{u}(-T, \cdot)\|_{3/2} + \|\hat{\eta}^+ - \hat{u}(T, \cdot)\|_{3/2} \right. \\ &\quad \left. + \|\hat{u}(-T, \cdot)\|_{3/2} + \|\hat{u}(T, \cdot)\|_{3/2} \right) \\ &\leq c_1 (\|D_0 \hat{u}\|_{1,2} + \|\hat{\eta}^- - \hat{u}(-T, \cdot)\|_{3/2} + \|\hat{\eta}^+ - \hat{u}(T, \cdot)\|_{3/2}) \end{aligned}$$

The first inequality in (4.94) follows by summing (4.91) and (4.92). The second inequality follows from trace inequality

$$\|\hat{u}(-T, \cdot)\|_{3/2} + \|\hat{u}(T, \cdot)\|_{3/2} \leq c\|\hat{u}\|_{2,2}$$

and inequalities (4.91) and (4.93). Thus we have proved that $d\mathcal{F}_\xi^T(0)$ is injective and has closed range as it satisfies the inequality (4.94). We prove that it is surjective. Suppose that there exist a vector $(\hat{v}, \zeta^+, \zeta^-)$ orthogonal to $\text{Im}(d\mathcal{F}_\xi^T(0))$. Taking $\hat{u} = 0$ and varying $\hat{\eta}^\pm$ we get $\zeta^+ \in E^+$, $\zeta^- \in E^-$. It follows from Theorem 3.1.6 that the operator

$$\hat{u} \mapsto \left(D_0 \hat{u}, \pi^+(\hat{u}(-T, \cdot)), \pi^-(\hat{u}(T, \cdot)) \right)$$

is bijective. Hence, for given ζ^\pm there exists unique $\hat{u} \in \text{Ker}(D_0)$ such that $\pi^\pm(\hat{u}(\mp T, \cdot)) = \zeta^\pm$. For such \hat{u} as $(\hat{v}, \zeta^+, \zeta^-)$ is orthogonal to $d\mathcal{F}_\xi^T(0)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)$ it follows

$$\langle \hat{u}(-T, \cdot), \zeta^+ \rangle + \langle \hat{u}(T, \cdot), \zeta^- \rangle = \|\zeta^+\|_{3/2}^2 + \|\zeta^-\|_{3/2}^2 = 0.$$

Hence $\zeta^+ = \zeta^- = 0$. From the surjectivity of the operator D_0 it follows that $\hat{v} = 0$. \square

Step B: *There exists a constant \tilde{r}_1 such that for all $T, \xi \in B_{\rho_0}(0)$, for all $\eta^\pm \in E^\pm$ and $u \in H_{bc}^2(Z^T)$ with:*

$$\|u\|_{2,2} < \tilde{r}_1 \tag{4.94}$$

the operator $d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)$ is bijective and

$$\|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)^{-1}\| \leq C_1 = \frac{6c_1}{5}, \tag{4.95}$$

where c_1 is the constant from step A.

Proof.

$$\begin{aligned} & \left\| \left(d\mathcal{F}_\xi^T(u, \eta^-, \eta^+) - d\mathcal{F}_\xi^T(0) \right) \left(\hat{\eta}^-, \hat{\eta}^+, \hat{u} \right) \right\| = \left\| \left(D_u \hat{u} - D_0 \hat{u}, 0, 0 \right) \right\| \\ & \leq c\|u\|_{2,2}\|\hat{u}\|_{2,2} \\ & \leq c\|u\|_{2,2}(\|\hat{u}\|_{2,2} + \|\hat{\eta}^-\|_{3/2} + \|\hat{\eta}^+\|_{3/2}) \end{aligned} \tag{4.96}$$

The first inequality in (4.96) follows from the inequality (4.83). Let $\tilde{r}_1 = \frac{1}{6c_1c}$, where c_1 is the constant of the step A and c as in (4.96), and suppose that $\|u\|_{2,2} < \tilde{r}_1$. We have that $\|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+) - d\mathcal{F}_\xi^T(0)\| \leq \frac{1}{6c_1}$ and hence

$$\|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+) \cdot d\mathcal{F}_\xi^T(0)^{-1} - 1\| \leq \frac{1}{6}, \quad \forall T \geq T_0 \quad (4.97)$$

From (4.97) it follows that $d\mathcal{F}_\xi^T(u, \eta^-, \eta^+) \cdot d\mathcal{F}_\xi^T(0)^{-1}$ is invertible and thus also $d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)$. Let $L = \|\hat{u}\|_{2,2} + \|\hat{\eta}^-\|_{3/2} + \|\hat{\eta}^+\|_{3/2}$, we have

$$\begin{aligned} L &\leq c_1 \|d\mathcal{F}_\xi^T(0)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)\| \\ &\leq c_1 \left(\|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)\| + \|(d\mathcal{F}_\xi^T(u, \eta^-, \eta^+) - d\mathcal{F}_\xi^T(0))(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)\| \right) \\ &\leq c_1 \left(\|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)\| + \frac{1}{6c_1} L \right) \end{aligned}$$

From the previous inequality we get

$$\|\hat{u}\|_{2,2} + \|\hat{\eta}^-\|_{3/2} + \|\hat{\eta}^+\|_{3/2} \leq \frac{6c_1}{5} \|d\mathcal{F}_\xi^T(u, \eta^-, \eta^+)(\hat{u}, \hat{\eta}^-, \hat{\eta}^+)\|. \quad (4.98)$$

□

Step C: The requirements 1) and 2) of Lemma 4.5.7 are satisfied for $r_1 = \frac{\tilde{r}_1}{2}$ and C_1 as in step B.

Proof. If $\|u\| < r_1$ we have that

- $d\mathcal{F}_\xi^T(u, 0, 0)$ is bijective and $\|d\mathcal{F}_\xi^T(u, 0, 0)^{-1}\| \leq C_1 = \frac{6c_1}{5}$
- For all v such that $\|u - v\|_{2,2} < r_1$ and for all $\eta^- \in E^-$, $\eta^+ \in E^+$ we have

$$\begin{aligned} \|d\mathcal{F}_\xi^T(u, 0, 0) - d\mathcal{F}_\xi^T(v, \eta^-, \eta^+)\| &\leq \left(\|d\mathcal{F}_\xi^T(u, 0, 0) - d\mathcal{F}_\xi^T(0)\| \right. \\ &\quad \left. + \|d\mathcal{F}_\xi^T(v, \eta^-, \eta^+) - d\mathcal{F}_\xi^T(0)\| \right) \\ &\leq \frac{1}{6c_1} + \frac{1}{6c_1} = \frac{1}{3c_1} < \frac{1}{2C_1}. \end{aligned}$$

□

Lemma 4.5.8. *There exists $\rho > 0$ and $T_1 > 0$ such that for every $T \geq T_1$ the following holds*

i) For every $\xi = (\xi^+, \xi^-) \in B_\rho(0)$ there exists unique $(u^T(\xi), \eta_-^T(\xi), \eta_+^T(\xi)) \in H_{bc}^2(Z^T) \times E^- \times E^+$ such that

$$\mathcal{F}_\xi^T(u^T, \eta_-^T, \eta_+^T) = 0.$$

ii) The maps $\eta^T = (\eta_-^T, \eta_+^T)$ converge exponentially to 0

$$\|\eta^T(\xi)\| \leq ce^{-\mu T},$$

for all $\xi \in B_\rho(0)$.

iii) The maps η^T converge also in C^1 norm exponentially to 0, i.e. for all $\xi \in B_\rho(0)$ the following inequality holds

$$\|d\eta^T(\xi)\| \leq ce^{-\mu T}, \quad (4.99)$$

where $\|d\eta^T(\xi)\| := \sup_{\|\hat{\xi}\|_{3/2} \leq 1} \|d\eta^T(\xi)\hat{\xi}\|_{3/2}$.

Step I) Proof of parts i) and ii) of Lemma 4.5.8

Proof. Let $\rho = \min\{\frac{r_1}{4c_0}, \rho_0\}$ where r_1 is the constant of Lemma 4.5.7, c_0 the constant of (4.80) in Step 1 and ρ_0 is the constant from Step 3. Let $(\xi^+, \xi^-) \in B_\rho(0) \subset E$. Then there exist holomorphic curves $u_\mp \in H_{bc}^2(Z^\pm)$ such that

$$u_-(0, \cdot) = \xi^+ + f^-(\xi^+), \quad u_+(0, \cdot) = \xi^- + f^+(\xi^-)$$

It follows from Step 1 and 2 in 4.5.5, as the mapping \mathcal{F}^∞ satisfies the requirements of the inverse function theorem, that $(0, 0, \xi^+, \xi^-) \in B_\rho(\mathcal{F}^\infty(0, 0)) \subset \mathcal{F}^\infty(B_{2c_0\rho}(0, 0))$, thus

$$\|u_+\|_{2,2} + \|u_-\|_{2,2} < 2c_0\rho \leq \frac{r_1}{2}.$$

We construct a pregluing map v_T as follows. Let $\beta(s) : \mathbb{R} \rightarrow [0, 1]$ be a smooth cut-off function such that

$$\beta(s) = \begin{cases} 1 & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 1 \end{cases}$$

Suppose that $\|\beta\|_{C^2} < 2$. Take

$$v_T = v_T(\xi) = \beta(s)u_-(s+T, t) + (1 - \beta(s))u_+(s-T, t).$$

The pregluing map v_T satisfies the following:

$$\begin{aligned} \|v_T\|_{2,2} &= \|\beta(s)u_+(s+T, t) + (1 - \beta(s))u_-(s-T, t)\|_{2,2} \\ &\leq \|\beta(s)u_+(s+T, t)\|_{2,2} + \|(1 - \beta(s))u_-(s-T, t)\|_{2,2} \\ &\leq \|\beta\|_{C^2}(\|u_+\|_{2,2} + \|u_-\|_{2,2}) < r_1 \end{aligned}$$

and also

$$v_T(\mp T, \cdot) = \xi^\pm(\cdot) + f^\mp(\xi^\pm).$$

Thus we have

$$\mathcal{F}_\xi^T(v_T, 0, 0) = (w_T, 0, 0), \quad (4.100)$$

where $w_T(s, t) = \bar{\partial}_{J_t, X_t} v_T$. As u_+ and u_- are the solutions of $\bar{\partial}_{J_t, X_t} u = 0$ we have

$$w_T(s, t) = \begin{cases} 0 & \text{if } s \leq -1 \\ 0 & \text{if } s \geq 1. \end{cases}$$

Hence, w_T is nonzero only on the interval $[-1, 1]$, but from exponential decay of holomorphic curves we know that u^\pm and all of their derivatives decay exponentially, thus

$$\|w_T\|_{1,2} \leq ce^{-\mu(T-1)}. \quad (4.101)$$

The constructed pregluing map v_T satisfies the assumptions a) and b) of Lemma 4.5.7. As $\mathcal{F}_\xi^T(v_T, 0, 0) = (w_T, 0, 0)$ and $\|w_T\|_{1,2}$ decays exponentially for some T_1 sufficiently large and $T \geq T_1$ we have

$$\|\mathcal{F}_\xi^T(v_T, 0, 0)\| = \|w_T\|_{1,2} \leq \frac{r_1}{2C_1},$$

where C_1 is the constant in Lemma 4.5.7. From the inverse function theorem we have

$$B_{\frac{r_1}{2C_1}}(\mathcal{F}_\xi^T(v_T, 0, 0)) \subset \mathcal{F}_\xi^T(B_{r_1}(v_T, 0, 0)).$$

Therefore there exist unique $(\eta_T^-(\xi), \eta_T^+(\xi), u_T(\xi)) \in B_{r_1}(v_T, 0, 0)$ such that

$$\mathcal{F}_\xi^T(u^T, \eta_-^T, \eta_+^T) = (0, 0, 0). \quad (4.102)$$

The last equality proves part i) of Lemma 4.5.8. From the inverse function theorem it follows also

$$\|u^T - v_T\|_{2,2} + \|\eta_-^T\|_{3/2} + \|\eta_+^T\|_{3/2} \leq 2C_1\|w_T\|_{1,2} \leq ce^{-\mu T}. \quad (4.103)$$

Hence we have

$$\|\eta_T^-(\xi)\|_{3/2} + \|\eta_T^+(\xi)\|_{3/2} \leq ce^{-\mu T}. \quad (4.104)$$

□

Step II) Proof of *iii*)

Proof. We finish the proof of Lemma 4.5.8 by proving part *iii*). Let $v_T(\xi)$ be the pregluing map. From the definition of the map \mathcal{F}^T we have

$$E^- \ni \eta_-^T(\xi) = (u^T(\xi) - v_T(\xi))(-T, \cdot), \quad E^+ \ni \eta_+^T(\xi) = (u^T(\xi) - v_T(\xi))(T, \cdot) \quad (4.105)$$

Let

$$\hat{u}_T = du^T(\xi)(\hat{\xi}), \quad \hat{v}_T = dv_T(\xi)(\hat{\xi}), \quad \hat{\eta}^T = d\eta^T(\xi)(\hat{\xi}).$$

Then

$$\hat{\eta}_-^T = \pi^-(\hat{\eta}^T) = (\hat{u}_T - \hat{v}_T)(-T, \cdot), \quad \hat{\eta}_+^T = \pi^+(\hat{\eta}^T) = (\hat{u}_T - \hat{v}_T)(T, \cdot)$$

As $\|u_T\|_{2,2}$ is sufficiently small, the following inequality holds for all $\xi \in H_{bc}^2(Z^T)$

$$\|\xi\|_{2,2} \leq c' \left(\|D_{u^T}(\xi)\|_{1,2} + \|\pi^+(\xi(-T, \cdot))\|_{3/2} + \|\pi^-(\xi(T, \cdot))\|_{3/2} \right). \quad (4.106)$$

The previous inequality follows from Steps 1 and 2 in 4.5.5. Substituting $\xi = \hat{u}_T - \hat{v}_T$ in (4.106) we get

$$\begin{aligned} \|\hat{u}_T - \hat{v}_T\|_{2,2} &\leq c' \left(\|D_{u^T}(\hat{u}_T - \hat{v}_T)\|_{1,2} + \|\pi^+(\hat{\eta}_-^T)\|_{3/2} + \|\pi^-(\hat{\eta}_+^T)\|_{3/2} \right) \\ &= c' \|D_{u^T}(\hat{u}_T - \hat{v}_T)\|_{1,2} \\ &= c' \|D_{u^T}(\hat{v}_T)\|_{1,2}, \end{aligned} \quad (4.107)$$

where the last equality holds as u^T satisfies $\bar{\partial}_{J_t, X_t} u^T = 0$ and hence $D_{u^T}(\hat{u}_T) = 0$. From the trace inequality and (4.107) we have

$$\|\hat{\eta}_-^T\|_{3/2} + \|\hat{\eta}_+^T\|_{3/2} \leq c \|D_{u^T}(\hat{v}_T)\|_{1,2}. \quad (4.108)$$

We shall prove that the right hand side of (4.108) decays exponentially. Notice that

$$\|D_{u^T} \hat{v}_T\|_{1,2} \leq \left(\overbrace{\|D_{v_T} \hat{v}_T\|_{1,2}}^I + \overbrace{\|(D_{u^T} - D_{v_T}) \hat{v}_T\|_{1,2}}^{II} \right)$$

Then

$$II = \|(D_{u^T} - D_{v_T}) \hat{v}_T\|_{1,2} \leq c \|u^T - v_T\|_{2,2} \|\hat{v}_T\|_{2,2} \leq c e^{-\delta T} \|\hat{v}_T\|_{2,2}, \quad (4.109)$$

the penultimate inequality follows from (4.83) and the last inequality follows from the inequality (4.103). As $\hat{v}_T = \beta du_+(\xi^+) \hat{\xi}^+ + (1 - \beta) du_-(\xi^-) \hat{\xi}^-$, we have

$$\|\hat{v}_T\|_{2,2} \leq 2 \left(\overbrace{\|du_+(\xi^+) \hat{\xi}^+\|_{2,2}}^{w_+} + \overbrace{\|du_-(\xi^-) \hat{\xi}^-\|_{2,2}}^{w_-} \right) \quad (4.110)$$

Both w_{\pm} are in the kernels of the operators $D_{u^{\pm}}$ and they satisfy

$$\begin{aligned} \|w_{\pm}\|_{2,2} &\leq c' \left(\|D_{u^{\pm}} w_{\pm}\|_{1,2} + \|\pi^{\pm}(w_{\pm}(0, \cdot))\|_{3/2} \right) \\ &\leq c' \|\pi^{\pm}(w_{\pm}(0, \cdot))\|_{3/2} \\ &\leq c \|\hat{\xi}^{\pm}\|_{3/2}. \end{aligned} \quad (4.111)$$

From the inequalities (4.109), (4.110), (4.111) we have

$$II \leq ce^{-\mu T} (\|\hat{\xi}^+\|_{3/2} + \|\hat{\xi}^-\|_{3/2}). \quad (4.112)$$

For the I term we have

$$I = \|D_{v_T} \hat{v}_T\|_{H^1([-T, T] \times [0, 1])} = \|D_{v_T} \hat{v}_T\|_{H^1([-1, 1] \times [0, 1])} \leq c \|\hat{v}_T\|_{H^2([-1, 1] \times [0, 1])}$$

and the penultimate inequality holds as

$$\hat{v}_T(s, t) = \begin{cases} du_+(\xi)(\hat{\xi}^+) = w_+(s + T, t), & (s, t) \in [-T, -1) \times [0, 1] \\ du_-(\xi)(\hat{\xi}^-) = w^-(s - T, t), & (s, t) \in (1, T] \times [0, 1] \end{cases}$$

and $D_{u^{\pm}} w^{\pm} = 0$, hence $D_{v_T} \hat{v}_T = 0$ on the $([-T, -1) \cup (1, T]) \times [0, 1]$. As w^{\pm} are in the kernel of the operators $D_{u^{\pm}}$ they will decay exponentially. In [21] Lemma 3.1 was proved that

$$\|w^+(s, \cdot)\|_{L^2((0, 1))} \leq c \|w^+(0, \cdot)\|_{L^2((0, 1))} e^{-\mu s}.$$

Thus it follows that $\|w^+\|_{L^2([s, +\infty) \times [0, 1])}$ decays exponentially. The exponential decay of $\|w^+\|_{W^{2,2}([s, +\infty) \times [0, 1])}$ follows from the following inequality

$$\|w^+\|_{W^{k,2}([s, +\infty) \times [0, 1])} \leq c \left(\|D_{u^+}(w^+)\|_{W^{k-1,2}([s-1, +\infty) \times [0, 1])} + \|w^+\|_{W^{k-1,2}([s-1, +\infty) \times [0, 1])} \right).$$

This inequality follows from Lemma C.1 in [21].

□

4.6 Appendix

4.6.1 The spaces H^s and H_0^s and $H_{00}^{1/2}$

We shall mention some important properties of these interpolation spaces. For more detailed exposition of these facts we refer to [15]. Let $I = [0, 1]$ we define the space $H^s(I)$ as the interpolation space

$$H^s(I) = [H^k(I), L^2(I)]_\theta, \quad k(1 - \theta) = s.$$

Analogously are defined the interpolation spaces $H^s(\mathbb{R}^n)$. For $0 < s < 1$ a function $u \in H^s(\mathbb{R}^n) = [H^1(\mathbb{R}^n), L^2(\mathbb{R}^n)]_{1-s}$ is characterized by the property that $u \in L^2(\mathbb{R}^n)$ and

$$\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty. \quad (4.113)$$

This can be proved using the Fourier transform. The analogous holds for functions $u \in H^s(I)$, $0 < s < 1$. Namely, a function $u \in H^s(I)$ if $u \in L^2(I)$ and

$$\iint_{I \times I} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dx dy < +\infty. \quad (4.114)$$

This follows from the fact that the restriction map $\text{Rest} : H^s(\mathbb{R}) \rightarrow H^s(I)$ is linear, bounded surjective map and it has bounded right inverse. Its inverse, i.e. the extension map

$$\text{Ext} : H^s(I) \rightarrow H^s(\mathbb{R})$$

is given by reflection in local coordinate charts.

Remark 4.6.1. The extension by 0 is not a bounded linear map on $H^{1/2}(I)$. Take for example $u = 1$. This function is an element of $H^{1/2}(0, 1)$ as the integral (4.114) is 0. But the extended function

$$\tilde{u}(x) = \begin{cases} 1, & x \in (0, 1) \\ 0, & x \notin (0, 1) \end{cases}$$

is not an element of $H^{1/2}(\mathbb{R})$, as the integral (4.113) diverges.

Definition 4.6.2. The space $H_0^s(I)$ is defined as the closure of $C_c^\infty(I)$ in the $H^s(I)$ -norm.

Theorem 4.6.3. (Extension by 0) *The map*

$$u \mapsto \tilde{u} = \text{extension of } u \text{ by } 0 \text{ outside of } I$$

is a continuous mapping of $H^s(I) \rightarrow H^s(\mathbb{R})$ iff $0 \leq s < 1/2$. The same mapping is a continuous mapping of

$$H_0^s(I) \rightarrow H^s(\mathbb{R})$$

if $s > 1/2$ and $s \neq \text{integer} + 1/2$.

From the previous theorem we see that the spaces H_0^s change their behavior exactly for $s = n + \frac{1}{2}$, $n \in \mathbb{N}_0$. In the next proposition we prove that the spaces $H_0^{1/2}(0, 1)$ and $H^{1/2}(0, 1)$ are the same.

Proposition 4.6.4. *The space $C_c^\infty((0, 1))$ is dense in $H^{1/2}((0, 1))$, hence*

$$H_0^{1/2}((0, 1)) = H^{1/2}((0, 1)).$$

Proof. The proof follows from the following facts:

Fact A: There exist a sequence of smooth functions $f_n : \mathbb{R}^2 \rightarrow \mathbb{R}$ with $f_n = 1$ on some neighborhood of 0, but $\|f_n\|_{H^1(\mathbb{R}^2)} \rightarrow 0$, $n \rightarrow \infty$.

Fact B: Let $g(x) = 1$ on $(-1, 1)$. There exists a sequence of smooth functions g_n such that

- $g_n(x) = 0$ on some open neighborhood U_n of 0.
- $\|g_n(x) - 1\|_{H^{1/2}(-1, 1)} \xrightarrow{n \rightarrow \infty} 0$

Fact C: Any polynomial $P_k(x) = a_k x^k + \cdots + a_1 x + a_0$, $a_i \in \mathbb{R}$, $i = 0, \dots, k$ can be approximated in $H^{1/2}(0, 1)$ norm by $C_c^\infty(0, 1)$ functions.

Proof of Fact A : Let $0 < \delta < \epsilon$, define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ as

$$f(z) = \begin{cases} \frac{\ln \epsilon - \ln |z|}{\ln \epsilon - \ln \delta}, & \delta \leq |z| \leq \epsilon \\ 1, & |z| < \delta \\ 0, & |z| > \epsilon. \end{cases}$$

Then obviously $f \in C(\mathbb{R}^2)$ and $\text{supp}(f) \subset B_\epsilon(0)$ and $|f'(z)| = \frac{1}{|z| \ln(\epsilon/\delta)}$ on $\delta \leq |z| \leq \epsilon$. We want to estimate $\|f\|_{H^1(\mathbb{R}^2)}$. First we have

$$\begin{aligned} I_{\epsilon, \delta} &= \int_{\mathbb{R}^2} |f'|^2 dx dy = \int_{\delta \leq |z| \leq \epsilon} |f'|^2 dx dy \\ &= \int_{\delta \leq |z| \leq \epsilon} \frac{dx dy}{|z|^2 \ln^2(\epsilon/\delta)} = \int_{\delta}^{\epsilon} \int_0^{2\pi} \frac{r d\phi dr}{r^2 \ln^2(\epsilon/\delta)} = \frac{2\pi}{\ln^2(\epsilon/\delta)} \int_{\delta}^{\epsilon} \frac{dr}{r} = \frac{2\pi}{\ln(\epsilon/\delta)} \end{aligned}$$

Thus, if $\delta \rightarrow 0$ much faster than ϵ we have that $I_{\epsilon,\delta} \rightarrow 0$. To obtain $\|f\|_{L^2} \rightarrow 0$ we need to take $\epsilon \rightarrow 0$. Take $\epsilon_n = \frac{1}{n}$ and $\delta_n = \frac{1}{n^2}$ and take the corresponding functions f_n , constructed as above. This is a sequence of continuous functions which satisfy

- $f_n(0) = 1$
- $f_n \xrightarrow{H^1(\mathbb{R}^2)} 0, n \rightarrow \infty$.

To get a sequence of smooth functions that satisfy the same, use mollifiers to smoothen f_n .

Proof of Fact B: It is enough to take $\tilde{g}_n = f_n(x, 0)$, where f_n is the sequence of smooth functions as in the proof of Fact A. Let $g_n(x) = 1 - \tilde{g}_n(x)$. Then

$$\|1 - g_n(x)\|_{H^{1/2}(-1,1)} = \|\tilde{g}_n(x)\|_{H^{1/2}(-1,1)} \stackrel{tr}{\leq} \|f_n\|_{H^1(\mathbb{R}^2)},$$

as $\lim_{n \rightarrow \infty} \|f_n\|_{H^1(\mathbb{R}^2)} = 0$ we get

$$\lim_{n \rightarrow \infty} \|1 - g_n(x)\|_{H^{1/2}(0,1)} = 0. \quad (4.115)$$

Proof of Fact C: Let $h_n(x) = g_n(x) \cdot g_n(1 - x)|_{(0,1)} \in C_c^\infty(0, 1)$, where g_n is the sequence as in the proof of Fact B. The sequence $h_n(x) \xrightarrow{H^{1/2}} 1, n \rightarrow \infty$. One can easily check that for every $m \in \mathbb{N}$

$$C_c^\infty(0, 1) \ni x^m h_n(x) \xrightarrow{H^{1/2}} x^m, n \rightarrow \infty \quad (4.116)$$

As polynomials are dense in $H^{1/2}((0, 1))$ ($C(\overline{\Omega})$ is dense in $H^{1/2}(\Omega)$) we get that $C_c^\infty(0, 1)$ is dense in $H^{1/2}(0, 1)$. \square

Thus, we see that there is no difference between $H^{1/2}$ and $H_0^{1/2}$ and that the extension by 0 is not a bounded linear map. There are 1/2 interpolation spaces that allow the extension by 0 and they are called Lions- Magenes spaces. These spaces are exactly those spaces on which our Hilbert manifold of paths $\mathcal{P}^{3/2}$, defined in (4.34), is modeled. We discuss them in more details.

4.6.2 Lions- Magenes interpolations spaces $H_{00}^{1/2}$

For $W = H_0^1((0, 1)) = \{\xi \in H^1((0, 1)) : \xi(i) = 0, i = 0, 1\}$ and $H = L^2((0, 1))$ we define *Lions- Magenes* interpolation space

$$H_{00}^{1/2}((0, 1)) := [W, H]_{1/2} \quad (4.117)$$

We have seen in Theorem 3.4.8 that $[W, H]_{1/2}$ can be seen as the trace space of some Hilbert space or the domain $\text{Dom}(\sqrt{A})$, where A is the operator as in the definition 3.4.4. To define the space $H_{00}^{1/2}((0, 1))$ one can take for example $A = \partial_t : H_0^{1/2} \rightarrow L^2$. In the next proposition we give another equivalent interpretation of this space.

Proposition 4.6.5. *A function $u \in H^{1/2}((0, 1))$ is an element of the space $H_{00}^{1/2}((0, 1)) = [W, H]_{1/2}$ if and only if*

$$\int_0^1 \frac{u^2(y)}{d(y, \partial I)} dy = \int_0^{1/2} \frac{u^2}{y} dy + \int_{1/2}^1 \frac{u^2(y)}{1-y} dy < +\infty \quad (4.118)$$

and the norm

$$\|u\|_{H_{00}^{1/2}} = \left(\|u\|_{H^{1/2}}^2 + \int_0^1 \frac{u^2(y)}{d(y, \partial I)} dy \right)^{1/2} \quad (4.119)$$

is equivalent to the interpolation norm.

Proof. Let $u \in H_{00}^{1/2} = [W, H]_{1/2}$. The extension by 0 is a bounded linear operator

$$\begin{aligned} \text{Ext}_0 : H_0^1((0, 1)) &\rightarrow H^1(\mathbb{R}) \\ \text{Ext}_0 : L^2((0, 1)) &\rightarrow L^2(\mathbb{R}). \end{aligned}$$

From the interpolation theorem it follows that the extension by 0 is a bounded linear operator

$$\text{Ext}_0 : H_{00}^{1/2}((0, 1)) \rightarrow H^{1/2}(\mathbb{R}).$$

Let $\tilde{u}(x)$ be the extended function i.e.

$$\tilde{u}(x) = \begin{cases} u(x), & x \in (0, 1) \\ 0, & x \in \mathbb{R} \setminus (0, 1) \end{cases}$$

As $\|\tilde{u}\|_{H^{1/2}(\mathbb{R})} \leq c\|u\|_{H_{00}^{1/2}}$ we have that

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^2} dx dy &= \int_{(0,1)} \int_{(0,1)} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^2} dx dy + \\ &+ 2 \underbrace{\int_{-\infty}^0 \int_0^1 \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^2} dx dy}_I + 2 \underbrace{\int_1^{+\infty} \int_0^1 \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^2} dx dy}_{II} \quad (4.120) \end{aligned}$$

Notice that

$$\begin{aligned}
 I &= \int_{-\infty}^0 \int_0^1 \frac{|u(x)|^2}{|x-y|^2} dx dy = \\
 &= \int_0^1 |u(x)|^2 dx \int_{-\infty}^0 \frac{dy}{|x-y|^2} = \int_0^1 |u(x)|^2 dx \int_0^{+\infty} \frac{dy}{(x+y)^2} \\
 &= \int_0^1 |u(x)|^2 dx \int_x^{+\infty} \frac{dt}{t^2} \\
 &= \int_0^1 \frac{|u(x)|^2}{x} dx. \tag{4.121}
 \end{aligned}$$

Similarly we get

$$II = \int_0^1 \frac{|u(x)|^2}{1-x} dx. \tag{4.122}$$

From the equations (4.120), (4.121) and (4.122) follows that the $u \in H^{1/2}((0, 1))$ and that the expression (4.118) is finite.

Assume now that the function $u \in H^{1/2}((0, 1))$ satisfies (4.118) and define $\tilde{u}(y)$ as follows

$$\tilde{u}(y) = \begin{cases} u(y), & y \in (0, 1) \\ 0, & y \in \mathbb{R} \setminus (0, 1) \end{cases}$$

From the above observation it follows that \tilde{u} is an element of $H^{1/2}(\mathbb{R})$ and thus there exists a function $v \in H^1(\mathbb{R}_+^2)$ such that $Tr(v) = v(0, y) = \tilde{u}(y)$. We can suppose that $v(x, y) = 0$ for $\|(x, y)\|$ big enough, otherwise take a function βv , where β is a suitable cut-off function.

There exists a locally Lipschitz homeomorphism $\Psi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}_+^2$, such that

$$\Psi(0 \times [0, 1]) = 0 \times [0, 1], \quad \Psi([0, +\infty) \times 0) = 0 \times (-\infty, 0]$$

and $\Psi([0, +\infty) \times 1) = 0 \times [1, +\infty)$. Let Ψ_1 be the mapping that stretches the strip, i.e. maps $[0, +\infty) \times [0, 1] \mapsto [0, +\infty) \times [-1, 1]$, $\Psi_1(x, y) = (x, 2(y - 1/2))$. Let $\Psi_2 : [0, +\infty) \times [-1, 1] \rightarrow [0, +\infty) \times \mathbb{R}$ be a mapping that maps vertical lines (x, y) , $0 \leq y \leq 1$ on the line through $A = (0, 1+x)$ and

$B = (0, x)$, fixing B , and similarly maps segment (x, y) , $-1 \leq y \leq 0$ on the line through $B = (0, x)$ and $C = (0, -(1+x))$, and it is given by

$$\Psi_2(x, y) = \begin{cases} (1-y)(x, 0) + y(0, 1+x) = ((1-y)x, y(1+x)), & 0 \leq y \leq 1 \\ (1+y)(x, 0) + y(0, 1+x) = ((1+y)x, y(1+x)), & -1 \leq y \leq 0 \end{cases}$$

Now $\Psi = \Psi_2 \circ \Psi_1$ is the desired map. The function $w = v \circ \Psi : [0, +\infty) \times [0, 1] \rightarrow \mathbb{R}$ and $w \in H^1([0, +\infty) \times [0, 1])$. As $v(y) = 0$ for $y \in (-\infty, 0) \cup (1, +\infty)$ we have that $w \in L^2([0, +\infty), H_0^1((0, 1)))$, and the function u satisfies

$$u = w|_{0 \times [0, 1]} \equiv u \in [L^2(0, 1), H_0^1((0, 1))]_{1/2} = H_{00}^{1/2}.$$

□

Remark 4.6.6. Notice that the space $H_{00}^{1/2}((0, 1))$ isn't closed in $H^{1/2}((0, 1)) = H_0^{1/2}((0, 1))$ and it has strictly finer topology!

4.6.3 The space $H_{bc}^{3/2}$

In this section we discuss the interpolation spaces which are relevant for our Hilbert manifold of paths $\mathcal{P}^{3/2}$ introduced in 4.2.9. Let

$$W = W_{bc}^{2,2}([0, 1]) := \left\{ \xi \in W^{2,2}([0, 1], \mathbb{R}^{2n}) \left| \begin{array}{l} \xi(i) \in \mathbb{R}^n \times \{0\}, \ i = 0, 1 \\ \partial_t \xi(i) \in \{0\} \times \mathbb{R}^n, \ i = 0, 1 \end{array} \right. \right\}.$$

and let

$$V = W_{bc}^{1,2}([0, 1]) := \left\{ \xi \in W^{1,2}([0, 1], \mathbb{R}^{2n}) \left| \xi(i) \in \mathbb{R}^n \times \{0\} \right. \right\}.$$

and let $H = L^2([0, 1], \mathbb{R}^{2n})$. Notice that

$$V = [W, H]_{1/2}.$$

We explain this fact in more details. Observe the following Hilbert space:

$$V_1 = \{ \xi \in W^{1,2}([0, 1], \mathbb{R}^{2n}) \left| \xi(0) \in \mathbb{R}^n \times \{0\}, \xi(1) \in \{0\} \times \mathbb{R}^n \right. \}.$$

The operator $A_1 = i\partial_t : V_1 \rightarrow H$ satisfies all the requirements of the Remark 3.4.6. Let $\Psi(t) \in U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R})$ be a smooth family such that $\Psi(0) = \text{Id}$ and $\Psi(1) : \mathbb{R}^n \times \{0\} \rightarrow \{0\} \times \mathbb{R}^n$. As $U(n)$ is connected

such $\Psi(t)$ exists. Notice that multiplication by Ψ defines an isometry between V and V_1 . As the operator $A_1 : V_1 \rightarrow H$ is bijective and self adjoint, then also its conjugate

$$A := \Psi^{-1}A_1\Psi = i(\partial_t + \Psi^{-1}\partial_t\Psi) : V \rightarrow H$$

is bijective and self adjoint with respect to the following scalar product

$$\langle \xi, \eta \rangle_H := \int_0^1 \langle \Psi(t)\xi(t), \Psi(t)\eta(t) \rangle dt.$$

Notice that $\text{Dom}(A) = V$ and $\text{Dom}(A^2) = W$. Thus we have

$$[W, H]_{1/2} = \text{Dom}(A) = V.$$

We define the Hilbert spaces $H_{bc}^{3/2}$ as the following interpolation space

$$H_{bc}^{3/2} = [W, V]_{1/2}.$$

From the previous discussion we have that $H_{bc}^{3/2} = [W, V]_{1/2} = [W, H]_{1/4} = \text{Dom}(|A|^{3/2})$, or analogously it can be defined as the set of all $\xi \in V$ such that

$$\begin{aligned} A\xi &\in [V, H]_{1/2} = [H^1([0, 1]) \times H_0^1([0, 1]), L^2([0, 1]) \times L^2([0, 1])]_{1/2} \\ &= H^{1/2}([0, 1], \mathbb{R}^n) \times H_{00}^{1/2}([0, 1], \mathbb{R}^n) \end{aligned}$$

If we write $\xi = (\xi_1, \xi_2)$, where ξ_1 and ξ_2 denote the first and last n coordinates of ξ , then $A\xi \in [V, H]_{1/2}$ implies that $\partial_t \xi_1 \in H_{00}^{1/2}([0, 1], \mathbb{R}^n)$ and $\partial_t \xi_2 \in H^{1/2}([0, 1], \mathbb{R}^n)$. Thus the Hilbert space $H_{bc}^{3/2}$ can be also given as

$$H_{bc}^{3/2} = \left\{ (\xi_1, \xi_2) \in H^1([0, 1], \mathbb{R}^n) \times H_0^1([0, 1], \mathbb{R}^n) \left| \begin{array}{l} \partial_t \xi_1 \in H_{00}^{1/2}, \\ \partial_t \xi_2 \in H^{1/2} \end{array} \right. \right\}.$$

Let I be an interval in \overline{R} and denote by $\mathcal{W}(I)$ the following space:

$$\mathcal{W}(I) = \{u | u \in L^2(I, W), \frac{\partial^2 u}{\partial s^2} \in L^2(I, H)\}$$

provided with the norm

$$\|u\|_{\mathcal{W}(I)}^2 = \left(\|u\|_{L^2(I, W)}^2 + \left\| \frac{\partial^2 u}{\partial s^2} \right\|_{L^2(I, H)}^2 \right).$$

The space $\mathcal{W}(I)$ is a Hilbert space and the space of smooth functions $\mathbb{C}_c^\infty(\overline{I}, W)$ is dense in $\mathcal{W}(I)$. It follows from intermediate derivative theorem

(Theorem 2.3 in [15]) that any function $u \in \mathcal{W}(I)$ satisfies $\partial_s u \in L^2(I, V)$ and that the mapping

$$\mathcal{W}(I) \rightarrow L^2(I, V), \quad u \mapsto \partial_s u$$

is continuous linear mapping. Thus we have that the norm $\|u\|_{\mathcal{W}(I)}$ is equivalent to the following norm

$$\|u\|_{\mathcal{W}(I)}^2 := \left(\|u\|_{\mathcal{W}(I)}^2 + \|\partial_s u\|_{L^2(I, V)}^2 \right).$$

With this norm, the Hilbert space $\mathcal{W}(I)$ is isometric with the following Hilbert space

$$W_{bc}^{2,2}(I \times [0, 1]) := \left\{ \xi \in W^{2,2}(I \times [0, 1], \mathbb{R}^{2n}) \mid \begin{array}{l} \xi(s, i) \in \mathbb{R}^n \times \{0\}, i = 0, 1 \\ \partial_t \xi(s, i) \in \{0\} \times \mathbb{R}^n, i = 0, 1 \end{array} \right\}.$$

It follows from Trace theorem (Theorem 3.2 in [15]), that for $u \in \mathcal{W}(I)$ and for some $s_0 \in I$ we have

$$u(s_0) \in [W, H]_{1/4} = [W, V]_{1/2}.$$

Thus the trace space of the Hilbert space $\mathcal{W}(I)$ is the Hilbert space $H_{bc}^{3/2}$. The following proposition is a corollary of the Theorem 8.3 in [15].

Proposition 4.6.7. *Let $W_{bc}^{2,2}(I \times [0, 1])$ and $H_{bc}^{3/2}$ be defined as above. Suppose that $s_0 \in I$ and denote with r the restriction map*

$$r : W_{bc}^{2,2}(I \times [0, 1]) \rightarrow H_{bc}^{3/2}, \quad r(\xi(s, t)) = \xi(s_0, t). \quad (4.123)$$

The linear map r is surjective and it has a continuous right inverse, i.e. there exists a continuous extension operator

$$Ext : H_{bc}^{3/2} \rightarrow W_{bc}^{2,2}(I \times [0, 1]). \quad (4.124)$$

4.7 Appendix-The Hessian of the symplectic action

This appendix explains why in symplectic Floer theory it's necessary to work with compatible (rather than tame) almost complex structure at least near the critical points. The asymptotic analysis requires that the Hessian of the symplectic action is self-adjoint operator for a suitable L^2 inner product. Theorem 4.7.2 below shows that this is only the case when the almost complex structure is chosen compatible with the symplectic form.

4.7.1 (Vector space setup). Let V be an even dimensional vector space and let a smooth family $J(t) \in \text{End}(V)$ satisfy $J(t)^2 = -\text{Id}$, $t \in [0, 1]$. Let Λ_0, Λ_1 be half dimensional subspaces and suppose that $A(t) \in \text{Aut}(V)$ is also a smooth family. Let

$$W_\Lambda^{1,2}([0, 1], V) := \{\xi \in W^{1,2}([0, 1], V) \mid \xi(i) \in \Lambda_i; i = 0, 1\}$$

Suppose that $\langle \cdot, \cdot \rangle_t$ is a smooth family of inner products on V and denote with

$$\langle \xi, \eta \rangle_{L^2} = \int_0^1 \langle \xi(t), \eta(t) \rangle_t dt$$

Observe the linear operator

$$\begin{aligned} D : W_\Lambda^{1,2}([0, 1], V) &\rightarrow L^2([0, 1], V) \\ (D\xi) &:= J(t)\dot{\xi}(t) + A(t)\xi(t) \end{aligned} \tag{4.125}$$

Theorem 4.7.2. Let $V, \Lambda_0, \Lambda_1, D, \langle \cdot, \cdot \rangle_t$ be as in 4.7.1. The operator D is self adjoint for $\langle \cdot, \cdot \rangle_{L^2}$ iff the following are satisfied

- i) $J(t)$ is compatible with $\langle \cdot, \cdot \rangle_t$, i.e. $\omega_t := \langle J(t)\cdot, \cdot \rangle_t$ is a non degenerate skew symmetric form, i.e. $\langle J(t)\cdot, \cdot \rangle_t = -\langle \cdot, J(t)\cdot \rangle_t$.
- ii) If Φ is a solution of

$$J\dot{\Phi} + A\Phi = 0, \Phi(0) = \text{Id}$$

then $\Phi(t)^*\omega_t = \omega_t(\Phi(t)\cdot, \Phi(t)\cdot) = \omega_0$ for all $t \in [0, 1]$, where

- iii) Λ_0 is Lagrangian for ω_0 and Λ_1 is Lagrangian for ω_1 .

Proof. In the special case $A = 0$ this theorem is equivalent to the following statement. Define the operator D by

$$D\xi := J(t)\xi$$

Then the operator D is self adjoint for $\langle \cdot, \cdot \rangle_{L^2}$ if and only if there exists a skew symmetric form $\omega : V \times V \rightarrow \mathbb{R}$ such that

- a) $\langle \cdot, \cdot \rangle_t = \omega(\cdot, J(t)(\cdot))$ for all t .
- b) Λ_0, Λ_1 are Lagrangian for ω .

We prove this special case in the next three steps. Assume without loss of generality $V = \mathbb{R}^{2n}$ and $\langle \xi, \eta \rangle_t = \xi^T Q(t) \eta$ for all t . Here $Q(t)$ are symmetric matrices. The operator D is self adjoint if and only if

$$0 = \langle \xi, D\eta \rangle_{L^2} - \langle D\xi, \eta \rangle_{L^2}$$

Then for all $\xi(t), \eta(t) \in V$ with $\xi(i), \eta(i) \in \Lambda_i$, $i = 0, 1$ we have

$$\begin{aligned} 0 &= \int_0^1 \left(\langle \xi, QJ\dot{\eta} \rangle - \langle J\dot{\xi}, Q\eta \rangle \right) dt \\ &= \int_0^1 \left\langle \xi, QJ\dot{\eta} + \frac{d}{dt}(J^T Q\eta) \right\rangle dt - \overbrace{\langle J(1)\xi(1), Q(1)\eta(1) \rangle + \langle J(0)\xi(0), Q(0)\eta(0) \rangle}^{\text{boundary cond.}} \\ &= \int_0^1 \langle \xi, (QJ + J^T Q)\dot{\eta} \rangle + \int_0^1 \left\langle \frac{d}{dt}(QJ)\xi, \eta \right\rangle + \text{boundary cond.} \end{aligned} \quad (4.126)$$

The second equality follows by partial integration. We prove that the equality (4.126) implies the following

1. $QJ + J^T Q = 0$.
2. QJ is constant.
3. $Q(0)J(0)\Lambda_0 \perp \Lambda_0$ and $Q(1)J(1)\Lambda_1 \perp \Lambda_1$

Step 1.

$$Q(t)J(t) + J^T(t)Q(t) = 0, \quad \forall t. \quad (4.127)$$

Proof. Suppose that $S(t_0) = (QJ + J^T Q)(t_0) \neq 0$ for some t_0 with $0 < t_0 < 1$. Choose ξ_0, η_0 such that

$$\langle \xi_0, S(t_0)\eta_0 \rangle = 1$$

Choose smooth functions α and β with compact support in $[t_0 - 2\varepsilon^2, t_0 + 2\varepsilon^2]$ such that $\beta(t_0 + t) = \frac{t}{\varepsilon}$, $|t| < \varepsilon^2$ and $\alpha(t_0 + t) = 1$ for $|t| < \varepsilon^2$. Let $\xi = \alpha\xi_0$ and $\eta = \beta\eta_0$. Thus it follows that

$$\int_0^1 \langle \partial_t(QJ)\xi, \eta \rangle_t \leq c\varepsilon^3$$

and

$$\int_0^1 \langle \xi, S\dot{\eta} \rangle \geq \delta\varepsilon^2$$

Thus it follows that the right hand side of (4.126) is different from zero, what is a contradiction. \square

Step 2. By Step 1 we have

$$\begin{aligned}
0 &= \langle \xi, D\eta \rangle_{L^2} - \langle D\xi, \eta \rangle_{L^2} \\
&= \langle Q(0)J(0)\xi(0), \eta(0) \rangle - \langle Q(1)J(1)\xi(1), \eta(1) \rangle + \int_0^1 \langle (\partial_t(QJ))\xi, \eta \rangle dt
\end{aligned} \tag{4.128}$$

and the previous equality holds for all ξ, η with $\xi(i), \eta(i) \in \Lambda_i$, thus obviously we have

- i) $\partial_t(QJ) = 0$
- ii) $Q(0)J(0)\Lambda_0 \perp \Lambda_0$ and $Q(1)\Lambda_1 \perp \Lambda_1$.

Step 3. In Steps 1 and 2 we have proved that

(a) $QJ = -(QJ)^T$ (Step 1)

(b) $QJ \equiv \text{constant}$ (Step 2)

(c) $Q(0)J(0)\Lambda_0 \perp \Lambda_0$ and $Q(1)\Lambda_1 \perp \Lambda_1$. Define

$$\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$$

by

$$\omega(\xi, \eta) := \langle QJ\xi, \eta \rangle$$

This is independent of t by (b) and skew symmetric by (a) and non degenerate by assumption. Moreover

•

$$\begin{aligned}
\omega(\xi, J(t)\eta) &= \langle Q(t)J(t)\xi(t), J(t)\eta(t) \rangle \stackrel{(a)}{=} -\langle J(t)^T Q(t)\xi(t), J(t)\eta(t) \rangle \\
&= -\langle Q(t)\xi(t), J(t)^2\eta(t) \rangle = \langle Q(t)\xi(t), \eta(t) \rangle = \langle \xi(t), \eta(t) \rangle_t
\end{aligned}$$

- $\omega(\xi_0, \eta_0) = \langle Q(0)J(0)\xi_0, \eta_0 \rangle = 0$ for all $\xi_0, \eta_0 \in \Lambda_0$ and similarly $\omega(\xi_1, \eta_1) = \langle Q(1)J(1)\xi_1, \eta_1 \rangle = 0$ for all $\xi_1, \eta_1 \in \Lambda_1$.

□

Step 4. Reducing the general case to the case $A = 0$.
 Conjugating the operator D with Φ , we reduce the general case to the case that $A = 0$. Then the proof follows from the first three steps. More precisely let $\xi = \Phi\tilde{\xi}$. Then we have

$$\begin{aligned}\tilde{D}\tilde{\xi} &= \Phi^{-1}D\Phi\tilde{\xi} \\ &= \Phi^{-1}(J\Phi\tilde{\xi} + J\dot{\Phi}\tilde{\xi} + A\Phi\tilde{\xi}) \\ &= \tilde{J}\tilde{\xi} + \underbrace{(\phi^{-1}J\dot{\Phi} + \Phi^{-1}A\Phi)}_{\tilde{A}}\tilde{\xi}\end{aligned}$$

Assume that $\tilde{A} = 0$. Thus we have that the operator

$$\tilde{D}\tilde{\xi} = \tilde{J}\tilde{\xi}$$

and it is self adjoint with respect to

$$\int_0^1 \langle \Phi\tilde{\xi}, \Phi\tilde{\eta} \rangle_t dt$$

It follows from the first three steps that there exist a two form $\omega : V \times V \rightarrow \mathbb{R}$ skew symmetric and non degenerate such that

$$\begin{aligned}\omega(\tilde{\xi}, \tilde{J}(t)\tilde{\eta}) &= \langle \Phi(t)\tilde{\xi}, \Phi(t)\tilde{\eta} \rangle_t \\ \omega(\tilde{\xi}, \Phi^{-1}J\Phi\tilde{\eta}) &= \langle \Phi\tilde{\xi}, \Phi\tilde{\eta} \rangle_t\end{aligned}$$

Thus we have that $\omega(\Phi^{-1}(t)\xi, \Phi^{-1}J(t)\eta) = \langle \xi, \eta \rangle_t$ and the smooth family of 2-forms $\omega_t := \omega(\Phi(t)^{-1}\cdot, \Phi(t)^{-1}\cdot)$ satisfies

$$\omega_t(\cdot, J(t)\cdot) = \langle \cdot, \cdot \rangle_t$$

We also have that

- 1) $\omega_t := \langle J(t)\cdot, \cdot \rangle_t = -\langle \cdot, J(t)\cdot \rangle_t$.
- 2) Λ_0 is Lagrangian for ω_0 and Λ_1 is Lagrangian for ω_1 .
- 3) If $J\dot{\Phi} + A\Phi = 0$, $\Phi(0) = \text{Id}$ then

$$\omega_t(\Phi(t)\xi, \Phi(t)\eta) = \omega_0(\xi, \eta)$$

4.7.3 (Question). Given Λ_0 and $\Lambda_1 \subset V$ and $J(t) \in \mathcal{J}(V)$ such that $J(0)\Lambda_0 \cap \Lambda_0$ and $J(1)\Lambda_1 \cap \Lambda_1$ and such that there exist a non degenerate skew form ω_0 such that $J(t)$ are tame with ω_0 for all time t . Does there exist a non degenerate skew form $\omega : V \times V \rightarrow \mathbb{R}$ such that

- 1) Λ_0, Λ_1 are Lagrangian for ω
- 2) $J(t)$ are compatible with ω for all t .

The answer to this question is no. More precisely there exists Λ_0, Λ_1 and $J(t)$ which satisfy the above condition and a non degenerate skew form ω_0 such that $J(t)$ are tamed by ω_0 for all t , but there doesn't exist ω such that $J(t)$ are compatible with ω for all t and such that $\Lambda_i, i = 0, 1$ are Lagrangian for ω .

4.7.4 (Counterexample.). Obviously, we cannot look for counterexample in the dimension 2, as here compatibility is the same as the tame condition. Thus we can suppose that $V = \mathbb{R}^4$. We take $\Lambda_0 = \mathbb{R}^2 \times \{0\}$ and $\Lambda_1 = \{0\} \times \mathbb{R}^2$. The standard symplectic form $\omega_0 = \sum_i dx_i \wedge dy_i$ is given by

$$\omega_0(z, z') = \langle x, y' \rangle - \langle y, x' \rangle$$

Let $B \in \mathbb{R}^{2 \times 2}$ be any matrix with $\text{Det}(B) \neq 0$. Observe the 2-form

$$\omega_B(z, z') := \langle x, By' \rangle - \langle y, Bx' \rangle$$

Notice that $\omega_B = \sum_{i,j} b_{ij} dx_i \wedge dy_j$. Any symplectic form in \mathbb{R}^4 can be written in the form

$$\omega = a dx_1 \wedge dx_2 + c dy_1 \wedge dy_2 + \omega_B$$

Notice that $\Lambda_0 = \mathbb{R}^2 \times \{0\}$ is Lagrangian if and only if $a = 0$ and similarly Λ_1 is Lagrangian if and only if $c = 0$. Thus both $\Lambda_i, i = 0, 1$ are Lagrangian if and only if $\omega = \omega_B$.

Let $A(t) \in \mathbb{R}^{2 \times 2}$ with $\text{Det}(A(t)) \neq 0$. Observe the following smooth family of $J(t)$.

$$J(t) := \begin{pmatrix} 0 & -A(t)^{-1} \\ A(t) & 0 \end{pmatrix}$$

Then $J(t)$ are compatible with ω_0 if and only if $A(t) = A(t)^T > 0$ and $J(t)$ are tame with ω_0 if and only if $A(t) + A(t)^T > 0$. Similarly we have that $J(t)$ are compatible with ω_B if and only if $BA = (BA)^T = A^T B^T > 0$. Consider the following three matrices $A_1 = \text{Id}$

$$A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \varepsilon \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}$$

These three matrices all satisfy the condition $A = A^T$ and they are all positive definite for small ϵ . Observe the fourth matrix A_4 given by

$$A_4 = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}$$

Then such matrix A_4 is obviously not symmetric but it satisfies the condition $A_4 + A_4^T > 0$.

Lemma 4.7.5. *Let A_i , $i = 1, 2, 3$ be as above and let $B \in \mathbb{R}^{2 \times 2}$ with $\det(B) \neq 0$ be such that*

$$BA_i = A_i^T B^T. \quad (4.129)$$

Then there exists a constant λ such that $B = \lambda \text{Id}$.

Proof. Notice first that in the case of $A_1 = \text{Id}$ the above condition is equivalent to

$$B = B^T = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

If the equality (4.129) is satisfied for A_1 and A_2 then it also holds for their difference, as well as for $M_1 = \frac{1}{\epsilon}(A_2 - A_1)$. From the equality $BM_1 = M_1^T B^T$ we obtain that $b_{12} = 0$. Similarly we have that the equality (4.129) also holds for the matrix $M_2 = \frac{1}{\epsilon}(A_3 - A_1)$. From the equality $BM_2 = M_2^T B^T$ we obtain $b_{11} = b_{22} = \lambda$.

□

Now we can finish the counterexample. Take a path $A(t)$ going through all these matrices (A_i , $i = 1, \dots, 4$) and such that $A(t) + A(t)^T > 0$. For example we can take $A(0) = A_1$, $A(\frac{1}{3}) = A_2$, $A(\frac{2}{3}) = A_3$ and $A(1) = A_4$. Then $J(t)$ is tamed by ω_0 for all t , but there doesn't exist a 2-form ω such that $J(t)$ are compatible with ω and that Λ_i , $i = 1, 2$ are Lagrangian subspaces. That follows from Lemma 4.7.5. If it would exist such ω , then we would have that $\omega = \omega_B$ and the matrix B satisfies $BA = A^T B^T$ for all $t \in [0, 1]$. Particularly the matrix B satisfies that condition for $t = \frac{i}{3}$, $i = 0, 1, 2$. Then it follows from Lemma 4.7.5 that $B = \lambda \text{Id}$, but the fourth matrix A_4 isn't symmetric and this is the contradiction!

Chapter 5

Applications in Lagrangian Floer homology

One of the three main technical ingredients in Lagrangian Floer theory is the Floer gluing theorem. (The other two are Floer–Gromov compactness and the linear elliptic Fredholm theory, including the Fredholm-index-equals-Maslov-index theorem). In chapter 4 we introduce a new approach to Lagrangian Floer gluing using nonlinear Hardy spaces. The purpose of the present chapter is to explain how the main result in chapter 4 implies the relevant gluing theorems in Lagrangian Floer theory (see Floer [4, 5] and Oh [17]) via intersection theory in a path space. More precisely, we prove the following results.

I: Boundary Map. The mod two count of solutions of the Floer equation for regular Floer data defines a map with square zero (Theorem 5.1.5).

II: Chain Map. The mod two count of solutions of the time dependent Floer equation, associated to a regular homotopy of Floer data, defines a homomorphism that intertwines the Floer boundary operators (Theorem 5.2.4).

III: Chain Homotopy Equivalence. The induced morphism on Floer homology in II is independent of the choice of the homotopy (Theorem 5.2.5).

IV: Catenation. Two composable morphisms on Floer homology as in III satisfy the composition rule under catenation of homotopies (Theorem 5.2.6).

The exposition here is restricted to monotone Lagrangian submanifolds with minimal Maslov numbers at least three. On the other hand we do not impose any restrictions on the fundamental groups or on the monotonicity factors of the Lagrangian submanifolds and hence it is necessary to work with Novikov rings. This chapter represents joint work with Prof. D. Salamon.

5.1 Floer Homology

This section is of expository nature. It discusses the basic setup of Lagrangian Floer theory in the monotone case (see [4, 5, 17]). More precisely, we consider the following setting.

(H) (M, ω) is a compact symplectic manifold without boundary and

$$L_0, L_1 \subset M$$

are compact Lagrangian submanifolds without boundary. For $i = 0, 1$ the pair (M, L_i) is monotone with minimal Maslov number at least three, i.e. for every smooth map $u : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L_i)$ the Maslov number $\mu(u)$ has absolute value at least three, and there is a constant $\tau_i > 0$ such that

$$\int_{\mathbb{D}} u^* \omega = \tau_i \mu(u)$$

every smooth map $u : (\mathbb{D}, \partial\mathbb{D}) \rightarrow (M, L_i)$.

5.1.1 (The Floer Equation and the Energy Identity). Fix a regular Hamiltonian function $H = \{H_t\}_{0 \leq t \leq 1} \in \mathcal{H}_{\text{reg}}(M, L_0, L_1)$ and a smooth family of ω -tame almost complex structures $J = \{J_t\}_{0 \leq t \leq 1} \in \mathcal{J}(M, \omega)$. The **Floer equation** has the form

$$\partial_s u + J_t(u)(\partial_t u - X_{H_t}(u)) = 0, \quad u(s, 0) \in L_0, \quad u(s, 1) \in L_1, \quad (5.1)$$

for a smooth map $u : \mathbb{R} \times [0, 1] \rightarrow M$. The **energy** of a solution u of (5.1) is defined by

$$E_H(u) := \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(|\partial_s u|_t^2 + |\partial_t u - X_{H_t}(u)|_t^2 \right) dt ds.$$

Here $\langle \xi, \eta \rangle_t := \frac{1}{2}(\omega(\xi, J_t \eta) + \omega(\eta, J_t \xi))$ denotes the Riemannian metric determined by ω and J_t . If the energy is finite then the limits

$$x^{\pm}(t) := \lim_{s \rightarrow \pm\infty} u(s, t) \in L_0 \cap L_1 \quad (5.2)$$

exist and belong to $\mathcal{C}(L_0, L_1; H)$ (see for example [21]). The convergence is with all derivatives, uniform in t , and exponential. For two solutions x^{\pm} of (4.3) denote the space of finite energy solutions of (5.1) and (5.2) by

$$\mathcal{M}(x^-, x^+; H, J) := \{u : \mathbb{R} \times [0, 1] \rightarrow M \mid (5.1), (5.2), E_H(u) < \infty\}.$$

(When $H = 0$ we abbreviate $\mathcal{M}(x^-, x^+; J) := \mathcal{M}(x^-, x^+; H, J)$.) Thus $\mathcal{M}(x^-, x^+; H, J)$ is the space of Floer trajectories from x^- to x^+ . Every finite energy solution of (5.1) and (5.2) satisfies the **energy identity**

$$E_H(u) = \int_{\mathbb{R} \times [0,1]} u^* \omega - \int_0^1 H_t(x^-(t)) dt + \int_0^1 H_t(x^+(t)) dt. \quad (5.3)$$

5.1.2 (Regular Pairs). A family $J \in \mathcal{J}(M, \omega)$ is called **regular for** L_0, L_1, H if every finite energy solution $u : \mathbb{R} \times [0, 1] \rightarrow M$ of (5.1) is regular in the sense that the linearized operator D_u is surjective. The set of regular families $J \in \mathcal{J}(M, \omega)$ will be denoted by $\mathcal{J}_{\text{reg}}(M, L_0, L_1, H)$. It is a residual subset of the space of $\mathcal{J}(M, \omega)$ (see Floer [4, 5]). A pair $(H, J) \in \mathcal{H}(M) \times \mathcal{J}(M, \omega)$ is called a **regular pair for** (L_0, L_1) if $H \in \mathcal{H}_{\text{reg}}(M, L_0, L_1)$ and $J \in \mathcal{J}_{\text{reg}}(M, L_0, L_1, H)$. The set of regular pairs for (L_0, L_1) is a residual subset of $\mathcal{H}(M) \times \mathcal{J}(M, \omega)$ and will be denoted by

$$\mathcal{HJ}_{\text{reg}}(M, L_0, L_1) := \left\{ (H, J) \mid \begin{array}{l} H \in \mathcal{H}_{\text{reg}}(M, L_0, L_1), \\ J \in \mathcal{J}_{\text{reg}}(M, L_0, L_1, H) \end{array} \right\}.$$

When $L_0 \overline{\cap} L_1$ and $H = 0$ we write $\mathcal{J}_{\text{reg}}(M, L_0, L_1) := \mathcal{J}_{\text{reg}}(M, L_0, L_1, H)$.

5.1.3 (Novikov Rings). Denote the **universal Novikov ring** with \mathbb{Z}_2 coefficients by

$$\Lambda := \left\{ \sum_{\varepsilon \in \mathbb{R}} \lambda_\varepsilon e^{-\varepsilon} \mid \lambda_\varepsilon \in \mathbb{Z}_2, \#\{\varepsilon \leq c \mid \lambda_\varepsilon \neq 0\} < \infty \forall c \in \mathbb{R} \right\}.$$

Each element of Λ can be thought of as a function $\mathbb{R} \rightarrow \mathbb{Z}_2 : \varepsilon \mapsto \lambda_\varepsilon$ with finite support over each half infinite interval $(-\infty, c]$. The universal Novikov ring is a field with multiplication $\lambda \lambda' := \sum_\varepsilon \sum_\delta \lambda_\delta \lambda'_\varepsilon e^{-\varepsilon}$.

5.1.4 (The Floer Chain Complex). Assume (M, L_0, L_1) satisfy (H) and let $(H, J) \in \mathcal{HJ}_{\text{reg}}(M, L_0, L_1)$. The **Floer chain complex** of L_0, L_1, H is the vector space over Λ generated by the solutions of (4.3). It is denoted by

$$\text{CF}_*(L_0, L_1; H) := \bigotimes_{x \in \mathcal{C}(L_0, L_1; H)} \Lambda x. \quad (5.4)$$

When $H = 0$ this chain complex is generated by the intersection points of L_0 and L_1 . Under our assumptions the space $\mathcal{M}(x^-, x^+; H, J)$ of Floer trajectories is a smooth manifold whose local dimension near $u \in \mathcal{M}(x^-, x^+; H, J)$ is the Viterbo–Maslov index $\mu_H(u)$ (see [4, 5, 19, 20, 27]). For every integer

$k \geq 0$ and every constant $\varepsilon > 0$ denote the space of Floer trajectories from x^- to x^+ with Viterbo–Maslov index k and energy ε by

$$\mathcal{M}_\varepsilon^k(x^-, x^+; H, J) := \{u \in \mathcal{M}(x^-, x^+; H, J) \mid \mu_H(u) = k, E_H(u) = \varepsilon\}.$$

This space is a k -dimensional manifold and (for $k > 0$) it carries a free and proper action of \mathbb{R} by translation. The quotient

$$\widehat{\mathcal{M}}_\varepsilon^k(x^-, x^+; H, J) := \mathcal{M}_\varepsilon^k(x^-, x^+; H, J)/\mathbb{R}$$

is a manifold of dimension $k - 1$. It follows from our hypotheses that

$$\sum_{\varepsilon \leq c} \#\widehat{\mathcal{M}}_\varepsilon^1(x^-, x^+; H, J) < \infty \quad \forall c > 0. \quad (5.5)$$

Define the operator $\partial = \partial^{H,J} : \text{CF}_*(L_0, L_1; H) \rightarrow \text{CF}_*(L_0, L_1; H)$ by

$$\partial^{H,J} x := \sum_{y \in \mathcal{C}(L_0, L_1; H)} \sum_{\varepsilon} \#\widehat{\mathcal{M}}_\varepsilon^1(x, y; H, J) e^{-\varepsilon} y \quad (5.6)$$

for $x \in \mathcal{C}(L_0, L_1; H)$. The following theorems assert that (5.6) is indeed a boundary operator and that the resulting Floer homology groups are invariant under Hamiltonian isotopy. The original proof by Floer [4, 5] was carried out under the assumption $\pi_2(M, L_i) = 0$. Floer's results were later extended to the monotone setting by Oh [17].

Theorem 5.1.5 (Boundary Operator). *Assume (H) and let (H, J) be a regular pair for (L_0, L_1) . Let $\partial^{H,J} : \text{CF}_*(L_0, L_1; H) \rightarrow \text{CF}_*(L_0, L_1; H)$ be defined by (5.6). Then $\partial^{H,J} \circ \partial^{H,J} = 0$.*

Proof. See Section 5.4. □

5.1.6 (Floer Homology). The homology

$$\text{HF}_*(L_0, L_1; H, J) := \frac{\ker \partial^{H,J}}{\text{im } \partial^{H,J}}$$

of the chain complex in Theorem 5.1.5 is called the **Floer homology group of (L_0, L_1)** associated to the regular pair (H, J) . The Floer homology of (L_0, L_1) is independent of the choice of the regular pair (H, J) up to canonical isomorphism.

Theorem 5.1.7 (Invariance). *Assume (H). There is a collection of isomorphisms $\Phi^{\beta\alpha} : \text{HF}_*(L_0, L_1; H^\alpha, J^\alpha) \rightarrow \text{HF}_*(L_0, L_1; H^\beta, J^\beta)$, one for any two regular pairs (H^α, J^α) and (H^β, J^β) , satisfying*

$$\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} = \Phi^{\gamma\alpha}, \quad \Phi^{\alpha\alpha} = \text{id}$$

for all $(H^\alpha, J^\alpha), (H^\beta, J^\beta), (H^\gamma, J^\gamma) \in \mathcal{HJ}_{\text{reg}}(M, L_0, L_1)$.

5.1.8 (Naturality). Assume (H) and let $(H, J) \in \mathcal{HJ}_{\text{reg}}(M, L_0, L_1)$. Let $[0, 1] \rightarrow \text{Diff}(M, \omega) : t \mapsto \psi_t$ be a Hamiltonian isotopy with corresponding family of Hamiltonian functions $K_t : M \rightarrow \mathbb{R}$, $0 \leq t \leq 1$, so that

$$\partial_t \psi_t = Y_t \circ \psi_t, \quad \iota(Y_t)\omega = dK_t. \quad (5.7)$$

We do not assume that ψ_0 is the identity. Let $u : \mathbb{R} \times [0, 1] \rightarrow M$ be a solution of (5.1) and define

$$\tilde{u}(s, t) := \psi_t^{-1}(u(s, t)), \quad \tilde{L}_0 := \psi_0^{-1}(L_0), \quad \tilde{L}_1 := \psi_1^{-1}(L_1),$$

and

$$\tilde{H}_t := (H_t - K_t) \circ \psi_t, \quad \tilde{X}_t := \psi_t^*(X_t - Y_t), \quad \tilde{J}_t := \psi_t^* J_t.$$

Then \tilde{H}_t generates the Hamiltonian isotopy $\tilde{\phi}_t := \psi_t^{-1} \circ \phi_t \circ \psi_0$ and \tilde{u} satisfies the Floer equation

$$\partial_s \tilde{u} + \tilde{J}_t(\tilde{u})(\partial_t \tilde{u} - \tilde{X}_t(\tilde{u})) = 0, \quad \tilde{u}(s, 0) \in \tilde{L}_0, \quad \tilde{u}(s, 1) \in \tilde{L}_1. \quad (5.8)$$

Thus pullback by ψ_t induces an isomorphism of Floer homology groups

$$\psi^* : \text{HF}_*(L_0, L_1; H, J) \rightarrow \text{HF}_*(\tilde{L}_0, \tilde{L}_1; \tilde{H}, \tilde{J}).$$

5.1.9 (Regular Floer Data and Naturality). Let (M, ω) be a compact symplectic manifold without boundary and denote by $\mathcal{F}_{\text{reg}} = \mathcal{F}_{\text{reg}}(M, \omega)$ the set of **regular Floer data** (L_0, L_1, H, J) , where $L_0, L_1 \subset M$ are Lagrangian submanifolds satisfying (H) and (H, J) is regular pair for (L_0, L_1) . The group $\mathcal{G} = \mathcal{G}(M, \omega)$ of Hamiltonian isotopies $\psi = \{\psi_t\}_{0 \leq t \leq 1}$ of (M, ω) (starting at any symplectomorphism) acts contravariantly on \mathcal{F}_{reg} via

$$\psi^*(L_0, L_1, H, J) := (\psi_0^{-1}(L_0), \psi_1^{-1}(L_1), \psi^* H, \psi^* J),$$

where

$$(\psi^* H)_t := (H_t - K_t) \circ \psi_t, \quad (\psi^* J)_t := \psi_t^* J_t,$$

and K_t is a family of Hamiltonian functions generating ψ_t via (5.7). More precisely, the homomorphism $\pi_2(M, L_i) \rightarrow \pi_2(M, \psi_i^{-1}(L_i)) : u \mapsto \psi_i^{-1} \circ u$ preserves the Maslov index and the symplectic area for $i = 0, 1$. Hence the pair $(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1))$ satisfies (H) whenever (L_0, L_1) does. Second, it follows from 5.1.8 that $(\psi^* H, \psi^* J) \in \mathcal{HJ}_{\text{reg}}(M, \psi_0^{-1}(L_0), \psi_1^{-1}(L_1))$ whenever $(H, J) \in \mathcal{HJ}_{\text{reg}}(M, L_0, L_1)$.

Theorem 5.1.10 (Lagrangian Seidel Homomorphism). *Fix a compact symplectic manifold (M, ω) without boundary. There is a collection of isomorphisms*

$$\psi^* : \mathrm{HF}_*(L_0, L_1; H, J) \rightarrow \mathrm{HF}_*(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1); \psi^*H, \psi^*J),$$

one for every $(L_0, L_1, H, J) \in \mathcal{F}_{\mathrm{reg}}$ and every $\psi \in \mathcal{G}$, satisfying the following.

(Functoriality) For all $(L_0, L_1, H, J) \in \mathcal{F}_{\mathrm{reg}}$ and $\phi, \psi \in \mathcal{G}$ we have

$$(\psi\phi)^* = \phi^* \circ \psi^* : \mathrm{HF}_*(L_0, L_1) \rightarrow \mathrm{HF}_*(\phi_0^{-1}(\psi_0^{-1}(L_0)), \phi_1^{-1}(\psi_1^{-1}(L_1))).$$

(Naturality) Let (L_0, L_1) satisfy (H), suppose $(H^\alpha, J^\alpha), (H^\beta, J^\beta)$ are regular pairs for (L_0, L_1) , and let $\psi \in \mathcal{G}$. Then the following diagram commutes

$$\begin{array}{ccc} \mathrm{HF}_*(L_0, L_1; H^\alpha, J^\alpha) & \xrightarrow{\psi^*} & \mathrm{HF}_*(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1); \psi^*H^\alpha, \psi^*J^\alpha) \\ \Phi^{\beta\alpha} \downarrow & & \downarrow \Phi^{\beta\alpha} \\ \mathrm{HF}_*(L_0, L_1; H^\beta, J^\beta) & \xrightarrow{\psi^*} & \mathrm{HF}_*(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1); \psi^*H^\beta, \psi^*J^\beta) \end{array} \quad (5.9)$$

(Isotopy) Let $(L_0, L_1, H^\alpha, J^\alpha) \in \mathcal{F}_{\mathrm{reg}}$ and $\phi, \psi \in \mathcal{G}$ such that

$$\phi_0^{-1}(L_0) = \psi_0^{-1}(L_0) =: \tilde{L}_0, \quad \phi_1^{-1}(L_1) = \psi_1^{-1}(L_1) =: \tilde{L}_1.$$

Define

$$(\tilde{H}^\beta, \tilde{J}^\beta) := (\phi^*H^\alpha, \phi^*J^\alpha), \quad (\tilde{H}^\gamma, \tilde{J}^\gamma) := (\psi^*H^\alpha, \psi^*J^\alpha).$$

Suppose ϕ is isotopic to ψ by a Hamiltonian isotopy $\{\psi_t^\lambda\}_{0 \leq t, \lambda \leq 1}$ that satisfies $\psi_0^\lambda(L_0) = \tilde{L}_0$ and $\psi_1^\lambda(L_1) = \tilde{L}_1$ for all λ . Then the following diagram commutes

$$\begin{array}{ccc} & \mathrm{HF}_*(L_0, L_1; H, J) & \\ \phi^* \swarrow & & \searrow \psi^* \\ \mathrm{HF}_*(\tilde{L}_0, \tilde{L}_1; \tilde{H}^\beta, \tilde{J}^\beta) & \xrightarrow{\Phi^{\gamma\beta}} & \mathrm{HF}_*(\tilde{L}_0, \tilde{L}_1; \tilde{H}^\gamma, \tilde{J}^\gamma) \end{array} \quad (5.10)$$

Here $\Phi^{\gamma\beta}$ is the isomorphism of Theorem 5.1.7.

Proof. See Section 5.2. □

Remark 5.1.11. (i) The Floer homology group $\mathrm{HF}_*(L_0, L_1)$ is a **connected simple system** in the sense of Conley, i.e. a small category with precisely one (iso)morphism between any two objects. The objects are regular pairs

(H, J) and the morphisms are the isomorphisms $\Phi^{\beta\alpha}$ of Theorem 5.1.7. More precisely one can define $\mathrm{HF}_*(L_0, L_1)$ as the set of all tuples $\{\xi^\alpha\}_\alpha$ of elements

$$\xi^\alpha \in \mathrm{HF}(L_0, L_1; H^\alpha, J^\alpha),$$

one for each regular pair $(H^\alpha, J^\alpha) \in \mathcal{HJ}_{\mathrm{reg}}(M, L_0, L_1)$, that satisfy

$$\xi^\beta = \Phi^{\beta\alpha} \xi^\alpha$$

for all α, β . This is a vector space over the universal Novikov ring Λ .

(ii) Theorem 5.1.10 shows that Floer homology is a contravariant functor from the category of Lagrangian pairs $L_0, L_1 \subset M$ satisfying (H), where the morphisms from (L_0, L_1) to $(\tilde{L}_0, \tilde{L}_1)$ are homotopy classes of Hamiltonian isotopies $\{\psi_t\}_{0 \leq t \leq 1}$ of (M, ω) satisfying

$$\psi_0^{-1}(L_0) = \tilde{L}_0, \quad \psi_1^{-1}(L_1) = \tilde{L}_1,$$

to the category of vector spaces over Λ .

(iii) The Floer chain complex is not equipped with a natural grading. A grading can be introduced via Seidel's notion of graded Lagrangian submanifolds, but we shall not discuss this here. If the Lagrangian submanifolds are oriented the Floer homology groups are graded modulo two by the intersection indices.

(iv) In favorable cases the moduli space $\mathcal{M}(x, y; H, J)$ of Floer trajectories are orientable and the Floer homology groups can be defined over the integers. However, we shall not discuss this here.

(v) The definition of the Floer homology groups of pairs of monotone Lagrangian submanifolds can sometimes be extended to the case where the minimal Maslov number is two. In this case, for $i = 0, 1$ and a generic almost complex structures J_i , there is a finite number of J_i -holomorphic Maslov index two discs in M with boundary in L_i , passing through a generic point in L_i . The parity $\varepsilon_i \in \{0, 1\}$ of this number is independent of J_i and of the chosen point in L_i . The Floer homology groups can still be defined if either a) $\varepsilon_0 = \varepsilon_1 = 0$ or b) $\varepsilon_0 + \varepsilon_1 = 0$ and the factors $\tau_0 = \tau_1$ in the definition of monotonicity agree.

(vi) One can define the Floer homology groups with coefficients in \mathbb{Z}_2 if in (H2) we have $\tau_0 = \tau_1 =: \tau$ and in addition the fundamental group of the space \mathcal{P} of paths from L_0 to L_1 is generated, modulo torsion, by $\pi_2(M, L_0)$ and $\pi_2(M, L_1)$. This the case whenever the image of the homomorphism $\pi_1(L_i) \rightarrow \pi_1(M)$ consists of torsion classes for $i = 0, 1$ (see Oh [17]). Here we do not make these assumptions and instead work with the Novikov ring Λ .

(vii) Let (S, σ) be a compact monotone symplectic manifold without boundary and let $\phi : S \rightarrow S$ be a symplectomorphism. Choose $M := S \times S$ with the product symplectic form $\omega := \pi_2^* \sigma - \pi_1^* \sigma$ and let $L_0 := \Delta$ (the diagonal in $M \times M$) and $L_1 := \text{graph}(\phi)$. Then there is a natural isomorphism

$$\text{HF}_*(\Delta, \text{graph}(\phi)) \cong \text{HF}_*(\phi).$$

(See [3] for the definition of $\text{HF}(\phi)$ and [1] for the isomorphism.)

5.1.12 (Outline). The proof of the identity $\partial^2 = 0$ in Theorem 5.1.5 is based on the study of the moduli space $\mathcal{M}^2(x, z; H, J)$ of index-2 solutions of the Floer equation (5.1) for two solutions x, z of (4.3). The 1-dimensional quotient space $\widehat{\mathcal{M}}^2(x, z; H, J)$ will in general not be compact. It can be compactified by including the zero dimensional product spaces $\widehat{\mathcal{M}}^1(x, y; H, J) \times \widehat{\mathcal{M}}^1(y, z; H, J)$ over all solutions y of (4.3). This is the content of the Floer gluing theorem (Section 5.4). In the present chapter we reduce Floer gluing to intersection theory in a path space (Theorem 4.1.8). This requires a monotonicity result for J -holomorphic curves with Lagrangian boundary conditions and a convergence theorem for a suitable family of nonlinear Hardy spaces.

5.2 Invariance

5.2.1 (Homotopy of Floer Data). Assume (H) and let

$$(H^\alpha, J^\alpha), (H^\beta, J^\beta) \in \mathcal{HJ}_{\text{reg}}(M, L_0, L_1)$$

be two regular pairs for (L_0, L_1) (see 5.1.2). Choose two Hamiltonian functions $F, G : \mathbb{R} \times [0, 1] \times M \rightarrow \mathbb{R}$ and a smooth family $J = \{J_{s,t}\}_{(s,t) \in \mathbb{R} \times [0,1]}$ of ω -tame almost complex structures on M such that

$$F_{s,0}|_{L_0} = \text{constant}, \quad F_{s,1}|_{L_1} = \text{constant} \quad (5.11)$$

for every $s \in \mathbb{R}$ and, for some $T > 0$, $F_{s,t} = 0$ for $|s| \geq T$ and

$$G_{s,t} = \begin{cases} -H_t^\alpha, & \text{if } s \leq -T, \\ -H_t^\beta, & \text{if } s \geq T, \end{cases} \quad J_{s,t} = \begin{cases} J_t^\alpha, & \text{if } s \leq -T, \\ J_t^\beta, & \text{if } s \geq T. \end{cases} \quad (5.12)$$

Here we denote $F_{s,t} := F(s, t, \cdot)$ and $G_{s,t} := G(s, t, \cdot)$. Such a triple (F, G, J) is called a **homotopy from (H^α, J^α) to (H^β, J^β)** . Condition (5.11) guarantees that L_0 is invariant under the Hamiltonian isotopy generated by $F_{s,0}$ and L_1 is invariant under the Hamiltonian isotopy generated by $F_{s,1}$. Equivalently, $\widetilde{L}_0 := \mathbb{R} \times L_0$ and $\widetilde{L}_1 := \mathbb{R} \times L_1$ are Lagrangian submanifolds of

the symplectic manifold $\widetilde{M} := \mathbb{R} \times [0, 1] \times M$ with the symplectic form $\widetilde{\omega} := \omega - d\widetilde{M}(F ds + G dt) + c ds \wedge dt$ (where $c > \max(\partial_s G - \partial_t F + \{F, G\})$).

Associated to every such homotopy is the **time dependent Floer equation** for a smooth map $u : \mathbb{R} \times [0, 1] \rightarrow M$. It has the form

$$\begin{aligned} \partial_s u + X_{F_{s,t}}(u) + J_{s,t}(u) (\partial_t u + X_{G_{s,t}}(u)) &= 0, \\ u(s, 0) &\in L_0, \quad u(s, 1) \in L_1. \end{aligned} \quad (5.13)$$

For $s \in \mathbb{R}$ and $t \in [0, 1]$ denote by $|\xi|_{s,t} := \sqrt{\omega(\xi, J_{s,t}\xi)}$ the Riemannian metric associated to ω and $J_{s,t}$. The **energy** of a solution u of (5.13) is defined by

$$\begin{aligned} E_{F,G}(u) &:= \frac{1}{2} \int_{-\infty}^{\infty} \int_0^1 \left(|\partial_s u + X_{F_{s,t}}(u)|_{s,t}^2 + |\partial_t u + X_{G_{s,t}}(u)|_{s,t}^2 \right) dt ds \\ &= \int_{-\infty}^{\infty} \int_0^1 \omega(\partial_s u + X_{F_{s,t}}(u), \partial_t u + X_{G_{s,t}}(u)) dt ds. \end{aligned}$$

If u is a solution of (5.13) with finite energy, then the limits

$$x^\alpha(t) = \lim_{s \rightarrow -\infty} u(s, t), \quad x^\beta(t) = \lim_{s \rightarrow \infty} u(s, t) \quad (5.14)$$

exist and $x^\alpha \in \mathcal{C}(L_0, L_1; H^\alpha)$, $x^\beta \in \mathcal{C}(L_0, L_1; H^\beta)$. The convergence is uniform in t , with all derivatives, and exponential.

5.2.2 (Relative Symplectic Action). The **relative symplectic action** of a finite energy solution u of (5.13) and (5.14) is the topological invariant $\mathcal{A}_H(u) \in \mathbb{R}$, defined by

$$\mathcal{A}_H(u) := \int_{\mathbb{R} \times [0, 1]} u^* \omega - \int_0^1 H_t^\alpha(x^\alpha(t)) dt + \int_0^1 H_t^\beta(x^\beta(t)) dt. \quad (5.15)$$

It is related to the energy by the formula

$$\begin{aligned} E_{F,G}(u) &= \mathcal{A}_H(u) + \int_{-\infty}^{\infty} \int_0^1 \left(\partial_s G - \partial_t F + \{F, G\} \right)(u) dt ds \\ &\quad + \int_{-\infty}^{\infty} \left(F_{s,1}(u(s, 1)) - F_{s,0}(u(s, 0)) \right) ds. \end{aligned} \quad (5.16)$$

The two integrals on the right satisfy a uniform bound, independent of u . Note that the energy agrees with the relative symplectic action whenever $F_{s,t} = 0$ and $G_{s,t} = -H_t$ for all s and t .

5.2.3 (Regular Homotopies). A homotopy (F, G, J) of Floer data from (H^α, J^α) to (H^β, J^β) as in 5.2.1 is called **regular** if every finite energy solution of (5.13) is regular in the sense that the linearized operator is surjective. The existence of a regular homotopy follows from standard transversality theory for the Floer equation (see for example [8]). Fix a regular homotopy (F, G, J) from (H^α, J^α) to (H^β, J^β) . Then the space

$$\mathcal{M}(x^\alpha, x^\beta; F, G, J) := \{u : \mathbb{R} \times [0, 1] \rightarrow M \mid (5.13), (5.14), \mathcal{A}_H(u) < \infty\}$$

of all smooth finite energy solutions of (5.13) and (5.14) is a smooth manifold whose local dimension near u is given by a suitable Maslov index $\mu_H(u)$. The k -dimensional part of $\mathcal{M}(x^\alpha, x^\beta; F, G, J)$ with action equal to ε is denoted

$$\mathcal{M}_\varepsilon^k(x^\alpha, x^\beta; F, G, J) := \left\{ u \in \mathcal{M}(x^\alpha, x^\beta; F, G, J) \mid \begin{array}{l} \mu_H(u) = k \\ \mathcal{A}_H(u) = \varepsilon \end{array} \right\}.$$

In the regular case the Floer–Gromov compactness theorem asserts that the union of the spaces $\mathcal{M}_\varepsilon^0(x^\alpha, x^\beta; F, G, J)$ over all $\varepsilon \leq c$ is a finite set for all $x^\alpha \in \mathcal{C}(L_0, L_1; H^\alpha)$, $x^\beta \in \mathcal{C}(L_0, L_1; H^\beta)$ and $c > 0$. Thus a regular homotopy (F, G, J) from (H^α, J^α) to (H^β, J^β) determines a linear operator $\Phi_{F,G,J}^{\beta\alpha} : \text{CF}_*(L_0, L_1; H^\alpha) \rightarrow \text{CF}_*(L_0, L_1; H^\beta)$, defined by

$$\Phi_{F,G,J}^{\beta\alpha} x^\alpha := \sum_{x^\beta \in \mathcal{C}(L_0, L_1; H^\beta)} \sum_{\varepsilon > 0} \#_2 \mathcal{M}_\varepsilon^0(x^\alpha, x^\beta; F, G, J) e^{-\varepsilon} x^\beta \quad (5.17)$$

for $x^\alpha \in \mathcal{C}(L_0, L_1; H^\alpha)$.

Theorem 5.2.4 (Chain Map). *Assume (H). Let (H^α, J^α) , (H^β, J^β) be regular pairs for (L_0, L_1) and let (F, G, J) be a regular homotopy from (H^α, J^α) to (H^β, J^β) . Then $\partial^{H^\beta, J^\beta} \circ \Phi_{F,G,J}^{\beta\alpha} = \Phi_{F,G,J}^{\beta\alpha} \circ \partial^{H^\alpha, J^\alpha}$.*

Proof. See Section 5.3 for a generalization. \square

Theorem 5.2.5 (Chain Homotopy Equivalence). *Assume (H) and let (H^α, J^α) , (H^β, J^β) , (F, G, J) be as in Theorem 5.2.4. Then the induced operator $\Phi^{\beta\alpha} : \text{HF}_*(L_0, L_1; H^\alpha, J^\alpha) \rightarrow \text{HF}_*(L_0, L_1; H^\beta, J^\beta)$ on Floer homology is independent of the choice of the regular homotopy (F, G, J) from (H^α, J^α) to (H^β, J^β) , used to define it.*

Proof. See Section 5.3 for a generalization. \square

Theorem 5.2.6 (Catenation). *Assume (H) and let (H^α, J^α) , (H^β, J^β) , (H^γ, J^γ) be regular pairs for (L_0, L_1) . Let $\Phi^{\beta\alpha}$, $\Phi^{\gamma\beta}$, $\Phi^{\gamma\alpha}$, $\Phi^{\alpha\alpha}$ be the operators on Floer homology defined in Theorems 5.2.4 and 5.2.5. Then $\Phi^{\alpha\alpha} = \text{id}$ and $\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} = \Phi^{\gamma\alpha}$.*

Proof. See Section 5.3 for a generalization. \square

5.2.7 (Naturality). Let F, G, J be as in Theorem 5.2.4 and suppose that $u : \mathbb{R} \times [0, 1] \rightarrow M$ is a finite energy solution of (5.13). Let $\mathbb{R} \times [0, 1] \rightarrow \text{Diff}(M, \omega) : (s, t) \mapsto \psi_{s,t}$ be a smooth family of Hamiltonian symplectomorphisms such that $\partial_s \psi_{s,t} = 0$ for $|s|$ sufficiently large and the Lagrangian submanifolds

$$\tilde{L}_0 := \psi_{s,0}^{-1}(L_0), \quad \tilde{L}_1 := \psi_{s,1}^{-1}(L_1)$$

are independent of s . Then there exist smooth families of Hamiltonian functions $A_{s,t}, B_{s,t} : M \rightarrow \mathbb{R}$ for $s \in \mathbb{R}$ and $0 \leq t \leq 1$ such that $A_{s,0}|_{L_0} = 0$, $A_{s,1}|_{L_1} = 0$, A has compact support, and

$$\partial_s \psi_{s,t} + X_{A_{s,t}} \circ \psi_{s,t} = 0, \quad \partial_t \psi_{s,t} + X_{B_{s,t}} \circ \psi_{s,t} = 0.$$

It follows that the function $\kappa(s, t) := \partial_s B_{s,t} - \partial_t A_{s,t} + \{A_{s,t}, B_{s,t}\}$ on M is constant for all s and t and vanishes for $|s|$ sufficiently large. Define

$$\tilde{u}(s, t) := \psi_{s,t}^{-1}(u(s, t)), \quad \tilde{J}_{s,t} := \psi_{s,t}^* J_{s,t},$$

$$\tilde{F}_{s,t} := (F_{s,t} - A_{s,t}) \circ \psi_{s,t}, \quad \tilde{G}_{s,t} := (G_{s,t} - B_{s,t}) \circ \psi_{s,t}.$$

Then \tilde{u} is a solution of (5.13) with F, G, J replaced by $\tilde{F}, \tilde{G}, \tilde{J}$. Moreover, we have

$$\partial_s \tilde{G} - \partial_t \tilde{F} + \{\tilde{F}, \tilde{G}\} = (\partial_s G - \partial_t F + \{F, G\}) \circ \psi - \kappa.$$

Hence the action of \tilde{u} agrees with the action of u and the energy of \tilde{u} agrees with the energy of u up to a global additive constant.

Corollary 5.2.8. Assume (H), let (H^α, J^α) be a regular pair for (L_0, L_1) , and let $\{\psi_t\}$ be a Hamiltonian isotopy such that

$$\psi_0(L_0) = L_0, \quad \psi_1(L_1) = L_1. \quad (5.18)$$

Let K_t be a family of Hamiltonian functions generating ψ_t via (5.7) and define

$$H_t^\beta := (H_t^\alpha - K_t) \circ \psi_t, \quad J_t^\beta := \psi_t^* J_t^\alpha.$$

Then (H^β, J^β) is a regular pair for (L_0, L_1) . If $\{\psi_t\}_{0 \leq t \leq 1}$ is Hamiltonian isotopic to the constant path $\psi_{0,t} = \text{id}$ subject to (5.18) then

$$\Phi^{\beta\alpha} = \psi^* : \text{HF}(L_0, L_1; H^\alpha, J^\alpha) \rightarrow \text{HF}(L_0, L_1; H^\beta, J^\beta).$$

Proof. By assumption, there exists a Hamiltonian isotopy

$$\mathbb{R} \times [0, 1] \times M \rightarrow M : (s, t, p) \mapsto \psi_{s,t}(p)$$

such that

$$\psi_{s,t} = \begin{cases} \text{id}, & \text{if } s \leq 0, \\ \psi_t, & \text{if } s \geq 1, \end{cases} \quad \psi_{s,0}(L_0) = L_0, \quad \psi_{s,1}(L_1) = L_1.$$

Choose Hamiltonian functions $A_{s,t}, B_{s,t} : M \rightarrow \mathbb{R}$ such that A has compact support, $A_{s,0}|_{L_0} = 0$, $A_{s,1}|_{L_1} = 0$, $B_{s,t} = -K_t$ for $s \geq 1$, and

$$\partial_s \psi_{s,t} + X_{A_{s,t}} \circ \psi_{s,t} = 0, \quad \partial_t \psi_{s,t} + X_{B_{s,t}} \circ \psi_{s,t} = 0.$$

Define

$$\tilde{F}_{s,t} := -A_{s,t} \circ \psi_{s,t}, \quad \tilde{G}_{s,t} := (-H_t^\alpha - B_{s,t}) \circ \psi_{s,t}, \quad \tilde{J}_{s,t} := \psi_{s,t}^* J_t^\alpha.$$

Then $(\tilde{F}, \tilde{G}, \tilde{J})$ is a homotopy from (H^α, J^α) to (H^β, J^β) as in 5.2.1, satisfying (5.11) and (5.12). Moreover, a smooth function $u : \mathbb{R} \times [0, 1] \rightarrow M$ is a solution of (5.1) with $H = H^\alpha$ and $J = J^\alpha$ if and only if the function

$$\tilde{u}(s, t) := \psi_{s,t}^{-1}(u(s, t))$$

is a solution of (5.13) with F, G, J replaced by $\tilde{F}, \tilde{G}, \tilde{J}$ (see 5.2.7). Since (H^α, J^α) is a regular pair for (L_0, L_1) , this implies that $(\tilde{F}, \tilde{G}, \tilde{J})$ is a regular homotopy from (H^α, J^α) to (H^β, J^β) . It also implies that the homomorphism on the Floer chain complex, induced by $(\tilde{F}, \tilde{G}, \tilde{J})$ agrees with ψ^* :

$$\Phi_{\tilde{F}, \tilde{G}, \tilde{J}}^{\beta\alpha} = \psi^* : \text{CF}_*(L_0, L_1; H^\alpha) \rightarrow \text{CF}_*(L_0, L_1; H^\beta).$$

Hence the assertion of Corollary 5.2.8 follows from Theorems 5.2.4 and 5.2.5. \square

Proof of Theorem 5.1.7. Let (H^α, J^α) and (H^β, J^β) be as in Theorem 5.2.4. By Theorems 5.2.4 and 5.2.5 there is a unique operator

$$\Phi^{\beta\alpha} : \text{HF}_*(L_0, L_1; H^\alpha, J^\alpha) \rightarrow \text{HF}_*(L_0, L_1; H^\beta, J^\beta).$$

By Theorem 5.2.6 this is an isomorphism with inverse $\Phi^{\alpha\beta}$ and these operators satisfy the requirements of Theorem 5.1.7. \square

Proof of Theorem 5.1.10. Suppose M, L_0, L_1 satisfy (H), let (H, J) be a regular pair for (L_0, L_1) , and let

$$\psi = \{\psi_t\}_{0 \leq t \leq 1}$$

be a Hamiltonian isotopy of (M, ω) , starting at any symplectomorphism. By 5.1.8, pullback defines an isomorphism

$$\psi^* : \mathrm{HF}_*(L_0, L_1; H, J) \rightarrow \mathrm{HF}_*(\psi_0^{-1}(L_0), \psi_1^{-1}(L_1); \psi^*H, \psi^*J).$$

That these isomorphisms satisfy the (*Functoriality*) condition is obvious from the definitions. That they satisfy the (*Naturality*) condition follows from 5.2.7 with

$$\psi_{s,t} := \psi_t, \quad A_{s,t} := 0, \quad B_{s,t} := -K_t.$$

That they satisfy the (*Isotopy*) condition with $\phi_t = \mathrm{id}$ follows from Corollary 5.2.8. That they satisfy the (*Isotopy*) condition in general follows from the special case $\phi_t = \mathrm{id}$ and the (*Functoriality*) condition. This proves Theorem 5.1.10. \square

5.3 Field Theory

It is convenient to extend the discussion of the previous section to a more general class of surfaces Σ with cylindrical ends (replacing the strip). In our exposition we follow the discussion in Seidel [26, pages 100-112]. The complex structure j on the surface will be fixed. Throughout we abbreviate

$$\mathbb{R}^+ := [0, \infty), \quad \mathbb{R}^- := (-\infty, 0].$$

Let (M, ω) be a compact symplectic manifold without boundary and

$$\mathcal{L} = \mathcal{L}(M, \omega)$$

be the set of all compact Lagrangian submanifolds $L \subset M$ without boundary such that (M, L) is monotone with minimal Maslov number at least three. Associated to (M, ω) is a category $\mathcal{L} = \mathcal{L}(M, \omega)$ defined as follows.

Definition 5.3.1 (The Category of Lagrangian pairs).

(OBJECTS) An **object** of \mathcal{L} is a map $I \rightarrow \mathcal{L} \times \mathcal{L} : i \mapsto (L_{i0}, L_{i1})$, defined on a finite set I .

(STRING COBORDISMS) Fix two objects $(L_0^\alpha, L_1^\alpha) = \{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}$ and $(L_0^\beta, L_1^\beta) = \{L_{i0}^\beta, L_{i1}^\beta\}_{i \in I^\beta}$ in $\mathcal{L}(M, \omega)$. A **string cobordism** from (L_0^α, L_1^α) to (L_0^β, L_1^β) is a tuple

$$(\Sigma, L) = (\Sigma, \{L_z\}_{z \in \partial\Sigma}, \{\iota_i^{\alpha,-}\}_{i \in I^\alpha}, \{\iota_i^{\beta,+}\}_{i \in I^\beta})$$

consisting of an oriented 2-manifold Σ with boundary, a locally constant map $\partial\Sigma \rightarrow \mathcal{L} : z \mapsto L_z$, and orientation preserving embeddings

$$\{\iota_i^{\alpha,-} : \mathbb{R}^- \times [0, 1] \rightarrow \Sigma\}_{i \in I^\alpha}, \quad \{\iota_i^{\beta,+} : \mathbb{R}^+ \times [0, 1] \rightarrow \Sigma\}_{i \in I^\beta},$$

that satisfy the following conditions.

(a) The images of the embeddings $\iota_i^{\alpha,-}$, $i \in I^\alpha$, and $\iota_i^{\beta,+}$, $i \in I^\beta$, are pairwise disjoint and their complement has compact closure.

(b) For $i \in I^\alpha$ and $s \leq 0$ we have $L_{\iota_i^{\alpha,-}(s,0)} = L_{i0}^\alpha$ and $L_{\iota_i^{\alpha,-}(s,1)} = L_{i1}^\alpha$.

(c) For $i \in I^\beta$ and $s \geq 0$ we have $L_{\iota_i^{\beta,+}(s,0)} = L_{i0}^\beta$ and $L_{\iota_i^{\beta,+}(s,1)} = L_{i1}^\beta$.

(MORPHISMS) Let $(\Sigma', L') = (\Sigma', \{L'_{z'}\}_{z' \in \partial\Sigma'}, \{\iota_i^{\alpha,-}\}_{i \in I^\alpha}, \{\iota_i^{\beta,+}\}_{i \in I^\beta})$ be another string cobordism from (L_0^α, L_1^α) to (L_0^β, L_1^β) . The string cobordisms (Σ, L) and (Σ', L') are called **equivalent** if there is an orientation preserving diffeomorphism $\phi : \Sigma \rightarrow \Sigma'$ such that

$$L'_{\phi(z)} = L_z, \quad \iota_i^{\alpha,-} = \phi \circ \iota_i^{\alpha,-}, \quad \iota_j^{\beta,+} = \phi \circ \iota_j^{\beta,+}$$

for $z \in \partial\Sigma$, $i \in I^\alpha$, $j \in I^\beta$. A **morphism in \mathcal{L} from (L_0^α, L_1^α) to (L_0^β, L_1^β)** is an equivalence class $[\Sigma, L]$ of string cobordisms.

(CATENATION) Let

$$(\Sigma^{\alpha\beta}, L^{\alpha\beta}) = \left(\Sigma^{\alpha\beta}, \{L_z^{\alpha\beta}\}_{z \in \partial\Sigma^{\alpha\beta}}, \{\iota_i^{\alpha,-}\}_{i \in I^\alpha}, \{\iota_i^{\beta,+}\}_{i \in I^\beta} \right) \quad (5.19)$$

be a string cobordism from (L_0^α, L_1^α) to (L_0^β, L_1^β) and

$$(\Sigma^{\beta\gamma}, L^{\beta\gamma}) = \left(\Sigma^{\beta\gamma}, \{L_z^{\beta\gamma}\}_{z \in \partial\Sigma^{\beta\gamma}}, \{\iota_i^{\beta,-}\}_{i \in I^\beta}, \{\iota_i^{\gamma,+}\}_{i \in I^\gamma} \right) \quad (5.20)$$

be a string cobordism from (L_0^β, L_1^β) to (L_0^γ, L_1^γ) . For $T > 0$ the **T -catenation** of these string cobordisms is the string cobordism

$$(\Sigma^{\alpha\beta}, L^{\alpha\beta}) \#_T (\Sigma^{\beta\gamma}, L^{\beta\gamma}) = \left(\Sigma_T^{\alpha\gamma}, \{L_z^{\alpha\gamma}\}_{z \in \partial\Sigma_T^{\alpha\gamma}}, \{\iota_i^{\alpha,-}\}_{i \in I^\alpha}, \{\iota_i^{\gamma,+}\}_{i \in I^\gamma} \right)$$

from $\{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}$ to $\{L_{i0}^\gamma, L_{i1}^\gamma\}_{i \in I^\gamma}$, defined as follows. The 2-manifold $\Sigma_T^{\alpha\gamma}$ is defined as the quotient

$$\begin{aligned}\Sigma_T^{\alpha\gamma} &:= \frac{\Sigma_{2T}^{\alpha\beta} \sqcup \Sigma_{2T}^{\beta\gamma}}{\equiv}, \\ \Sigma_{2T}^{\alpha\beta} &:= \Sigma^{\alpha\beta} \setminus \bigcup_{i \in I^\beta} \iota_i^{\beta,+}([2T, \infty) \times [0, 1]), \\ \Sigma_{2T}^{\beta\gamma} &:= \Sigma^{\beta\gamma} \setminus \bigcup_{i \in I^\beta} \iota_i^{\beta,-}((-\infty, -2T] \times [0, 1]),\end{aligned}\tag{5.21}$$

and the equivalence relation is given by $\iota_i^{\beta,+}(s, t) \equiv \iota_i^{\beta,-}(s - 2T, t)$ for $i \in I^\beta$, $0 < s < 2T$, $0 \leq t \leq 1$. The Lagrangian submanifolds $L_z^{\alpha\gamma} \subset M$ are

$$L_z^{\alpha\gamma} := \begin{cases} L_z^{\alpha\beta}, & \text{for } z \in \partial\Sigma^{\alpha\beta} \cap \Sigma_{2T}^{\alpha\beta}, \\ L_z^{\beta\gamma}, & \text{for } z \in \partial\Sigma^{\beta\gamma} \cap \Sigma_{2T}^{\beta\gamma}. \end{cases}\tag{5.22}$$

The equivalence class of $(\Sigma^{\alpha\beta}, L^{\alpha\beta}) \#_T (\Sigma^{\beta\gamma}, L^{\beta\gamma})$ is independent of T and depends only on the equivalence classes of $(\Sigma^{\alpha\beta}, L^{\alpha\beta})$ and $(\Sigma^{\beta\gamma}, L^{\beta\gamma})$.

(COMPOSITION) The **composition** in \mathcal{L} is defined by

$$[\Sigma^{\beta\gamma}, L^{\beta\gamma}] \circ [\Sigma^{\alpha\beta}, L^{\alpha\beta}] := [(\Sigma^{\alpha\beta}, L^{\alpha\beta}) \#_T (\Sigma^{\beta\gamma}, L^{\beta\gamma})].$$

5.3.2 (Floer Homology). Let $(L_0^\alpha, L_1^\alpha) = \{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}$ be an object in \mathcal{L} and fix a corresponding collection $(H^\alpha, J^\alpha) = \{H_i^\alpha, J_i^\alpha\}_{i \in I^\alpha}$ of regular pairs. Associated to these data is the Floer chain complex

$$\mathrm{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) := \bigotimes_{i \in I^\alpha} \mathrm{CF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$$

This is the vector space over Λ generated by the tuples $x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}$ with $x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$. The boundary operator

$$\partial^\alpha := \partial^{H^\alpha, J^\alpha} : \bigotimes_{i \in I^\alpha} \mathrm{CF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha) \rightarrow \bigotimes_{i \in I^\alpha} \mathrm{CF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$$

is induced by the boundary operators

$$\partial^{H_i^\alpha, J_i^\alpha} : \mathrm{CF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha) \rightarrow \mathrm{CF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$$

of Section 5.1. Its Floer homology group is denoted

$$\mathrm{HF}_*(L_0^\alpha, L_1^\alpha; H^\alpha, J^\alpha) := \frac{\ker \partial^\alpha}{\mathrm{im} \partial^\alpha} = \bigotimes_{i \in I^\alpha} \mathrm{HF}_*(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha, J_i^\alpha).$$

5.3.3 (Floer Data on String Cobordisms). Fix a string cobordism (Σ, L) from the object $(L_0^\alpha, L_1^\alpha) = \{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}$ in \mathcal{L} to $(L_0^\beta, L_1^\beta) = \{L_{i0}^\beta, L_{i1}^\beta\}_{i \in I^\beta}$. A set of Floer data on Σ is a triple (j, H, J) consisting of a complex structure j on Σ , a 1-form $H : T\Sigma \rightarrow \Omega^0(M)$, and a smooth family of ω -tame almost complex structures $J = \{J_z\}_{z \in \Sigma}$, satisfying the following conditions.

(a) $\iota_i^{\alpha,-}$ is holomorphic for $i \in I^\alpha$ and $\iota_i^{\beta,+}$ is holomorphic for $i \in I^\beta$.

(b) For $i \in I^\alpha$ there is a regular pair (H_i^α, J_i^α) for $(L_{i0}^\alpha, L_{i1}^\alpha)$ such that

$$(\iota_i^{\alpha,-})^* H = -H_{it}^\alpha dt, \quad J_{\iota_i^{\alpha,-}(s,t)} = J_{it}^\alpha \quad \text{for } s \leq 0, 0 \leq t \leq 1.$$

(c) For $i \in I^\beta$ there is a regular pair (H_i^β, J_i^β) for $(L_{i0}^\beta, L_{i1}^\beta)$ such that

$$(\iota_i^{\beta,+})^* H = -H_{it}^\beta dt, \quad J_{\iota_i^{\beta,+}(s,t)} = J_{it}^\beta \quad \text{for } s \leq 0, 0 \leq t \leq 1.$$

(d) The restriction $H_{z,\widehat{z}}|_{L_z}$ is constant for $z \in \partial\Sigma$ and $\widehat{z} \in T_z\partial\Sigma$.

The Floer data (j, H, J) are said to **connect** $(H^\alpha, J^\alpha) = \{H_i^\alpha, J_i^\alpha\}_{i \in I^\alpha}$ to $(H^\beta, J^\beta) = \{H_i^\beta, J_i^\beta\}_{i \in I^\beta}$. The tuple $\mathcal{S} := (\Sigma, L, j, H, J)$ is called a **framed string cobordism** from the tuple $\mathcal{L}^\alpha = (L_0^\alpha, L_1^\alpha, H^\alpha, J^\alpha)$ to the tuple $\mathcal{L}^\beta = (L_0^\beta, L_1^\beta, H^\beta, J^\beta)$.

5.3.4 (The Floer Equation on String Cobordisms). Let (L_0^α, L_1^α) and (L_0^β, L_1^β) be two objects in \mathcal{L} and (H^α, J^α) and (H^β, J^β) be two corresponding collections of regular pairs. Fix a string cobordism (Σ, L) from (L_0^α, L_1^α) to (L_0^β, L_1^β) , and a set of Floer data (j, H, J) on Σ from (H^α, J^α) to (H^β, J^β) . Associated to these data is the Floer equation

$$\bar{\partial}_{J,H}(u) := \frac{1}{2} \left(d_H u + J \circ (d_H u) \circ j \right) = 0, \quad u(z) \in L_z \text{ for } z \in \partial\Sigma, \quad (5.23)$$

for smooth maps $u : \Sigma \rightarrow M$. Here $d_H u \in \Omega^1(\Sigma, u^*TM)$ denotes the 1-form on Σ with values in the pullback tangent bundle of M , defined by

$$d_H u(z)\widehat{z} := du(z)\widehat{z} + X_{H_{z,\widehat{z}}}(u(z))$$

for $\widehat{z} \in T_z\Sigma$ and we abbreviate $(J \circ (d_H u) \circ j)(z)\widehat{z} := J_z(u(z))d_H u(z)j(z)\widehat{z}$. If $u : \Sigma \rightarrow M$ is a solution of (5.23) then the functions

$$u_i^{\alpha,-} := u \circ \iota_i^{\alpha,-} : \mathbb{R}^- \times [0, 1] \rightarrow M, \quad i \in I^\alpha,$$

satisfy the usual Floer equation (5.1) for the quadruple $(L_{i0}^\alpha, L_{i1}^\alpha, H_i^\alpha, J_i^\alpha)$. Similarly, the functions

$$u_i^{\beta,+} := u \circ \iota_i^{\beta,+} : \mathbb{R}^+ \times [0, 1] \rightarrow M, \quad i \in I^\beta,$$

satisfy (5.1) for the quadruple $(L_{i0}^\beta, L_{i1}^\beta, H_i^\beta, J_i^\beta)$.

5.3.5 (The Energy Identity for String Cobordisms). The energy of a solution $u : \Sigma \rightarrow M$ of (5.23) is defined by

$$E_H(u) := \frac{1}{2} \int_{\Sigma} |d_H u|_z^2 \, \text{dvol}_{\Sigma}$$

Here $\text{dvol}_{\Sigma} \in \Omega^2(\Sigma)$ is a volume form compatible with the orientation and $\langle \cdot, \cdot \rangle := \text{dvol}_{\Sigma}(\cdot, \cdot)$ is the associated Riemannian metric on Σ . The term on the right in (5.26) is the integral of the function $\Sigma \rightarrow \mathbb{R} : z \mapsto |d_H u(z)|_z^2$, where $|d_H u(z)|_z$ denotes the operator norm of $d_H u(z) : T_z \Sigma \rightarrow T_{u(z)} M$ with respect to the above metric on Σ and the Riemannian metric on M determined by J_z and ω . This integral is independent of the choice of dvol_{Σ} .

If a solution of (5.23) has finite energy $E_H(u) < \infty$ then the limits

$$x_i^{\alpha}(t) = \lim_{s \rightarrow -\infty} u_i^{\alpha,-}(s, t), \quad x_i^{\beta}(t) = \lim_{s \rightarrow \infty} u_i^{\beta,+}(s, t), \quad (5.24)$$

exist for $i \in I^{\alpha}$, respectively $i \in I^{\beta}$, the convergence in (5.24) is uniform in t , with all derivatives, and exponential, and $x_i^{\alpha} \in \mathcal{C}(L_{i0}^{\alpha}, L_{i1}^{\alpha}; H_i^{\alpha})$ for $i \in I^{\alpha}$ and $x_i^{\beta} \in \mathcal{C}(L_{i0}^{\beta}, L_{i1}^{\beta}; H_i^{\beta})$ for $i \in I^{\beta}$. The **relative symplectic action** of a solution $u : \Sigma \rightarrow M$ of (5.23) and (5.24) is the number

$$\mathcal{A}_H(u) := \int_{\Sigma} u^* \omega - \sum_{i \in I^{\alpha}} \int_0^1 H_{it}^{\alpha}(x_i^{\alpha}(t)) \, dt + \sum_{i \in I^{\beta}} \int_0^1 H_{it}^{\beta}(x_i^{\beta}(t)) \, dt. \quad (5.25)$$

It is related to the energy by

$$E_H(u) = \mathcal{A}_H(u) + \int_{\Sigma} u^* \Omega_H - \int_{\partial \Sigma} u^* H. \quad (5.26)$$

The 2-form $\Omega_H \in \Omega^2(\Sigma, \Omega^0(M))$ is the curvature of H , defined by

$$\Omega_{H,z}(\widehat{z}_1, \widehat{z}_2) := dH_z(\widehat{z}_1, \widehat{z}_2) + \{H_{z,\widehat{z}_1}, H_{z,\widehat{z}_2}\} \in \Omega^0(M), \quad \widehat{z}_1, \widehat{z}_2 \in T_z \Sigma.$$

The value of the first term on the right at a point $p \in M$ denotes the differential of the 1-form

$$T\Sigma \rightarrow \mathbb{R} : (z, \widehat{z}) \mapsto H_{z,\widehat{z}}(p)$$

and $\{F, G\} := \omega(X_F, X_G)$ is the Poisson bracket. The scalar differential forms $u^* \Omega_H \in \Omega^2(\Sigma)$ and $u^* H \in \Omega^1(\Sigma)$ are defined by evaluating at $u(z)$. Note that (5.16) is a special case of (5.26).

5.3.6 (Regular Floer Data on String Cobordisms). The Floer data (j, H, J) are called **regular for Σ, L** if every finite energy solution of (5.23) is regular in the sense that the linearized operator is surjective. In this case the tuple

$$\mathcal{S} := (\Sigma, L; j, H, J)$$

is called a **regular framed string cobordism** from $\mathcal{L}^\alpha = (L_0^\alpha, L_1^\alpha; H^\alpha, J^\alpha)$ to $\mathcal{L}^\beta = (L_0^\beta, L_1^\beta; H^\beta, J^\beta)$. The existence of regular Floer data on any string cobordism (Σ, L) and for any fixed complex structure j on Σ follows from the standard transversality arguments in Floer theory (see for example [8] and also [16]). Moreover, it suffices to perturb H in $U \times M$ for any given fixed nonempty open subset $U \subset \Sigma$ to achieve transversality.

5.3.7 (Morphisms on Floer Homology). Fix a regular set of Floer data (j, H, J) connecting (H^α, J^α) to (H^β, J^β) . Fix two tuples of critical points

$$x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}, \quad x^\beta = \{x_i^\beta\}_{i \in I^\beta}$$

with

$$x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha), \quad x_i^\beta \in \mathcal{C}(L_{i0}^\beta, L_{i1}^\beta; H_i^\beta).$$

Define

$$\mathcal{M}(x^\alpha, x^\beta; j, H, J) := \left\{ u : \Sigma \rightarrow M \mid (5.23), (5.24), \mathcal{A}_H(u) < \infty \right\}.$$

This space is a smooth manifold whose local dimension near u is given by a suitable Maslov index $\mu_H(u)$. The k -dimensional part of $\mathcal{M}(x^\alpha, x^\beta; j, H, J)$ with relative symplectic action ε is denoted

$$\mathcal{M}_\varepsilon^k(x^\alpha, x^\beta; j, H, J) := \left\{ u \in \mathcal{M}(x^\alpha, x^\beta; j, H, J) \mid \begin{array}{l} \mu_H(u) = k, \\ \mathcal{A}_H(u) = \varepsilon \end{array} \right\}. \quad (5.27)$$

In the regular case the Floer–Gromov compactness theorem asserts that the union of the spaces $\mathcal{M}_\varepsilon^0(x^\alpha, x^\beta; j, H, J)$ over all $\varepsilon \leq c$ is a finite set for all x^α and x^β and $c > 0$. Thus (j, H, J) determines a linear operator

$$\Phi_{\Sigma, L; j, H, J}^{\beta\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\beta, L_1^\beta; H^\beta),$$

defined by

$$\Phi_{\Sigma, L; j, H, J}^{\beta\alpha} x^\alpha := \sum_{x^\beta} \sum_{\varepsilon} \#_2 \mathcal{M}_\varepsilon^0(x^\alpha, x^\beta; j, H, J) e^{-\varepsilon} x^\beta. \quad (5.28)$$

Theorem 5.3.8 (Chain Map). *Let $\mathcal{S} := (\Sigma, L; j, H, J)$ be a regular framed string cobordism from $\mathcal{L}^\alpha = (L_0^\alpha, L_1^\alpha; H^\alpha, J^\alpha)$ to $\mathcal{L}^\beta = (L_0^\beta, L_1^\beta; H^\beta, J^\beta)$. Then*

$$\partial^\beta \circ \Phi_{\Sigma, L; j, H, J}^{\beta\alpha} = \Phi_{\Sigma, L; j, H, J}^{\beta\alpha} \circ \partial^\alpha.$$

Proof. See Section 5.4. \square

5.3.9 (Homotopy of Floer Data). Let (L_0^α, L_1^α) and (L_0^β, L_1^β) be two objects in \mathcal{L} and (H^α, J^α) and (H^β, J^β) be two corresponding collections of regular pairs. Let (Σ, L) be a string cobordism from (L_0^α, L_1^α) and (L_0^β, L_1^β) and let (j_0, H_0, J_0) and (j_1, H_1, J_1) be two regular sets of Floer data on Σ from (H^α, J^α) to (H^β, J^β) . Denote the corresponding chain homomorphism between the Floer complexes by

$$\Phi_0^{\beta\alpha}, \Phi_1^{\beta\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\beta, L_1^\beta; H^\beta).$$

Choose a smooth homotopy

$$\{j_\lambda, H_\lambda, J_\lambda\}_{0 \leq \lambda \leq 1}$$

of Floer data from (j_0, H_0, J_0) to (j_1, H_1, J_1) , for each λ connecting (H^α, J^α) to (H^β, J^β) . The homotopy can be chosen **regular** in the sense that the linearized operator for the one parameter Floer equation

$$\bar{\partial}_{J_\lambda, H_\lambda}(u) = 0, \quad u(z) \in L_z \text{ for } z \in \partial\Sigma, \quad (5.29)$$

is surjective. In this case the moduli space

$$\mathcal{M}_\varepsilon^k(x^\alpha, x^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) := \left\{ (\lambda, u) \left| \begin{array}{l} 0 \leq \lambda \leq 1, u : \Sigma \rightarrow M, \\ (5.29), (5.24), \mu_H(u) = k, \\ \mathcal{A}_{H_\lambda}(u) = \varepsilon \end{array} \right. \right\} \quad (5.30)$$

is a smooth manifold of dimension $k + 1$ for every tuple $x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}$ with $x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$ and every $x^\beta = \{x_i^\beta\}_{i \in I^\beta}$ with $x_i^\beta \in \mathcal{C}(L_{i0}^\beta, L_{i1}^\beta; H_i^\beta)$. Moreover, the usual Floer–Gromov compactness theorem asserts that for $k = -1$ the union of the moduli spaces $\mathcal{M}_\varepsilon^k(x^\alpha, x^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$ over all $\varepsilon \leq c$ is a finite set for all $c > 0$ and all x^α, x^β . Hence there is an operator

$$\Psi^{\beta\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\beta, L_1^\beta; H^\beta)$$

defined by

$$\Psi^{\beta\alpha} x^\alpha := \sum_{x^\beta} \sum_{\varepsilon} \# \mathcal{M}_\varepsilon^{-1}(x^\alpha, x^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) e^{-\varepsilon} x^\beta. \quad (5.31)$$

Theorem 5.3.10 (Chain Homotopy Equivalence). *The above operators $\Phi_0^{\beta\alpha}$, $\Phi_1^{\beta\alpha}$, and $\Psi^{\beta\alpha}$ satisfy the equation*

$$\Phi_1^{\beta\alpha} - \Phi_0^{\beta\alpha} = \partial^\beta \circ \Psi^{\beta\alpha} + \Psi^{\beta\alpha} \circ \partial^\alpha.$$

Proof. See Section 5.4. □

Corollary 5.3.11. *Let (Σ, L) , (j, H, J) , and $\Phi_{\Sigma, L; j, H, J}^{\beta\alpha}$ be as in Theorem 5.3.8. Then the induced homomorphism*

$$\Phi_{\Sigma, L}^{\beta\alpha} : \mathrm{HF}_*(L_0^\alpha, L_1^\alpha; H^\alpha, J^\alpha) \rightarrow \mathrm{HF}_*(L_0^\beta, L_1^\beta; H^\beta, J^\beta) \quad (5.32)$$

on Floer homology is independent of the choice of the regular Floer data from (H^α, J^α) to (H^β, J^β) , used to define it.

Proof. This follows immediately from Theorem 5.3.10. □

Theorem 5.3.12 (Catenation). *Fix a string cobordism $(\Sigma^{\alpha\beta}, L^{\alpha\beta})$ from the object (L_0^α, L_1^α) to (L_0^β, L_1^β) and a string cobordism $(\Sigma^{\beta\gamma}, L^{\beta\gamma})$ from (L_0^β, L_1^β) to (L_0^γ, L_1^γ) and denote by $(\Sigma^{\alpha\gamma}, L^{\alpha\gamma})$ their T -catenation (for any $T > 0$). Let $\Phi^{\beta\alpha}$, $\Phi^{\gamma\beta}$, $\Phi^{\gamma\alpha}$ be the operators on Floer homology associated to these cobordisms via Theorem 5.3.8 and Corollary 5.3.11. Then*

$$\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} = \Phi^{\gamma\alpha}. \quad (5.33)$$

Proof. See Section 5.4. □

Proof of Theorems 5.2.4, 5.2.5. Choosing

$$\Sigma := \mathbb{R} \times [0, 1]$$

we find that Theorem 5.2.4 is a special case of Theorem 5.3.8, and Theorem 5.2.5 is a special case of Corollary 5.3.11. □

5.3.13 (The Donaldson Category). Take $\Sigma = \Delta$ to be a connected genus zero surface with three cylindrical ends. Then the operator $\Phi_{\Sigma, L}^{\beta\alpha}$ in Theorem 5.3.8 and Corollary 5.3.11 defines a homomorphism

$$\mathrm{HF}_*(L_0, L_1) \otimes \mathrm{HF}_*(L_1, L_2) \rightarrow \mathrm{HF}_*(L_0, L_2).$$

This is the **Donaldson triangle product**. Associativity follows by splitting a genus zero surface with four cylindrical ends. This determines a category, where the objects are the monotone Lagrangian submanifolds $L \in \mathcal{L}(M, \omega)$ with minimal Maslov number at least three and the set of morphisms from L_0 to L_1 is the Floer homology group $\mathrm{HF}_*(L_0, L_1)$. Composition is given by the Donaldson triangle product.

5.4 Floer Gluing

Boundary Operator

In this section we prove Theorem 5.1.5. The proof relies on the following version of the Floer gluing theorem.

Theorem 5.4.1 (Floer Gluing/Boundary Operator). *Assume (H) and let (H, J) be a regular pair for (L_0, L_1) . Choose three Hamiltonian paths $x, y, z \in \mathcal{C}(L_0, L_1; H)$ and two Floer trajectories $u \in \mathcal{M}^1(x, y; H, J)$ and $v \in \mathcal{M}^1(y, z; H, J)$. Fix two real numbers $s_u, s_v \in \mathbb{R}$. Then there exist constants $T_0 > 0$ and $\delta_0 > 0$ and a smooth map*

$$(T_0, \infty) \rightarrow \mathcal{M}^2(x, z; H, J) : T \mapsto u_T \quad (5.34)$$

satisfying the following conditions.

(i) *The composition of (5.34) with the projection*

$$\mathcal{M}^2(x, z; H, J) \rightarrow \widehat{\mathcal{M}}^2(x, z; H, J)$$

is a diffeomorphism onto its image.

(ii) *The functions $(s, t) \mapsto u_T(s - T, t)$ converge to u and the functions $(s, t) \mapsto u_T(s + T, t)$ converge to v as T tends to infinity; in both cases the convergence is uniform with all derivatives on every compact subset of $\mathbb{R} \times [0, 1]$. Moreover,*

$$\lim_{T \rightarrow \infty} \sup_t \left(\sup_{s \leq 0} d(u_T(s - T, t), u(s, t)) + \sup_{s \geq 0} d(u_T(s + T, t), v(s, t)) \right) = 0.$$

(iii) *For every $T > T_0$ we have*

$$E_H(u_T) = E_H(u) + E_H(v).$$

(iv) *If $u' \in \mathcal{M}^2(x, z; H, J)$ satisfies*

$$E_H(u') = E_H(u) + E_H(v)$$

and

$$\inf_{s \in \mathbb{R}} \sup_{0 \leq t \leq 1} d(u'(s, t), u(s_u, t)) < \delta_0, \quad \inf_{s \in \mathbb{R}} \sup_{0 \leq t \leq 1} d(u'(s, t), v(s_v, t)) < \delta_0, \quad (5.35)$$

then u' agrees with u_T up to time shift for some $T > T_0$.

Proof. See Section 5.6. □

Proof of Theorem 5.1.5. Fix a pair $x, z \in \mathcal{C}(L_0, L_1; H)$ and a constant $\varepsilon > 0$. By Theorem 5.4.1 and the standard compactness theorem for Floer trajectories, the 1-dimensional moduli space

$$\widehat{\mathcal{M}}_\varepsilon^2(x, z; H, J) := \left\{ [u] \in \widehat{\mathcal{M}}_\varepsilon^2(x, z; J) \mid E_H(u) = \varepsilon \right\}$$

admits a compactification to a compact 1-manifold $\overline{\widehat{\mathcal{M}}}_\varepsilon^2(x, z; H, J)$ with boundary

$$\partial \overline{\widehat{\mathcal{M}}}_\varepsilon^2(x, z; H, J) = \bigcup_{y \in \mathcal{C}(L_0, L_1; H)} \bigcup_{0 < \delta < \varepsilon} \widehat{\mathcal{M}}_\delta^1(x, y; H, J) \times \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y, z; H, J).$$

Since every compact 1-manifold has an even number of boundary points, this implies

$$\sum_{y \in \mathcal{C}(L_0, L_1; H)} \sum_{0 < \delta < \varepsilon} \# \widehat{\mathcal{M}}_\delta^1(x, y; H, J) \cdot \# \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y, z; H, J) \in 2\mathbb{Z}$$

for every $\varepsilon > 0$ and every pair of intersection points $x, z \in L_0 \cap L_1$. This is equivalent to the formula

$$\partial^{H, J} \circ \partial^{H, J} = 0$$

and proves Theorem 5.1.5. \square

Chain Map

In this section we prove Theorem 5.3.8.

5.4.2. Let (M, ω) be a compact symplectic manifold. Fix two objects

$$(L_0^\alpha, L_1^\alpha) = \{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}, \quad (L_0^\beta, L_1^\beta) = \{L_{i0}^\beta, L_{i1}^\beta\}_{i \in I^\beta}$$

in $\mathcal{L}(M, \omega)$ and two collections

$$(H^\alpha, J^\alpha) = \{H_i^\alpha, J_i^\alpha\}_{i \in I^\alpha}, \quad (H^\beta, J^\beta) = \{H_i^\beta, J_i^\beta\}_{i \in I^\beta}$$

such that (H_i^α, J_i^α) is a regular pair for $(L_{i0}^\alpha, L_{i1}^\alpha)$ when $i \in I^\alpha$ and (H_i^β, J_i^β) is a regular pair for $(L_{i0}^\beta, L_{i1}^\beta)$ when $i \in I^\beta$. Let

$$(\Sigma, L) = \left(\Sigma, \{L_z\}_{z \in \partial \Sigma}, \{\iota_i^{\alpha, -}\}_{i \in I^\alpha}, \{\iota_i^{\beta, +}\}_{i \in I^\beta} \right)$$

be a string cobordism from (L_0^α, L_1^α) to (L_0^β, L_1^β) and let (j, H, J) be a regular set of Floer data on Σ from (H^α, J^α) to (H^β, J^β) .

Theorem 5.4.3 (Floer Gluing/Chain Map). *Let (Σ, L) and (j, H, J) be as in 5.4.2. Fix three tuples*

$$x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}, \quad y^\beta = \{y_i^\beta\}_{i \in I^\beta}, \quad z^\beta = \{z_i^\beta\}_{i \in I^\beta},$$

with $x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$ and $y_i^\beta, z_i^\beta \in \mathcal{C}(L_{i0}^\beta, L_{i1}^\beta; H_i^\beta)$, and let

$$u \in \mathcal{M}^0(x^\alpha, y^\beta; j, H, J), \quad v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta).$$

Thus $v = \{v_i\}_{i \in I^\beta}$ with $v_i \in \mathcal{M}(y_i^\beta, z_i^\beta; H_i^\beta, J_i^\beta)$ and there is an index $i_0 \in I^\beta$ such that $v_i(s, t) = y_i^\beta(t) = z_i^\beta(t)$ for $i \neq i_0$ and

$$\mu_H(v_{i_0}) = 1.$$

Fix any nonempty open set $W_0 \subset \Sigma \setminus \text{im } \iota_{i_0}^{\beta,+}$ and a real number $s_v \in \mathbb{R}$. Then there exist constants $T_0 > 0$ and $\delta_0 > 0$ and a smooth map

$$(T_0, \infty) \rightarrow \mathcal{M}^1(x^\alpha, z^\beta; j, H, J) : T \mapsto u_T \quad (5.36)$$

satisfying the following conditions.

- (i) *The map (5.36) is a diffeomorphism onto its image.*
- (ii) *The maps u_T converge to u as T tends to infinity and*

$$v_{i_0}(s, t) = \lim_{T \rightarrow \infty} u_T \left(\iota_{i_0}^{\beta,+}(s + T, t) \right)$$

for $s \in \mathbb{R}$ and $0 \leq t \leq 1$. In both cases the convergence is uniform with all derivatives on every compact subset of Σ , respectively $\mathbb{R} \times [0, 1]$. Moreover,

$$\lim_{T \rightarrow \infty} \left(\sup_{\Sigma \setminus \text{im } \iota_{i_0}^{\beta,+}} d(u_T, u) + \sup_{s \geq 0} \sup_{0 \leq t \leq 1} d(u_T \circ \iota_{i_0}^{\beta,+}(s + T, t), v_{i_0}(s, t)) \right) = 0.$$

- (iii) *$\mathcal{A}_H(u_T) = \mathcal{A}_H(u) + \mathcal{A}_H(v_{i_0})$ for every $T > T_0$.*
- (iv) *If $u' \in \mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$ satisfies*

$$\mathcal{A}_H(u') = \mathcal{A}_H(u) + \mathcal{A}_H(v_{i_0})$$

and

$$\sup_{W_0} d(u', u) < \delta_0, \quad \inf_{s \geq 0} \sup_{0 \leq t \leq 1} d(u' \circ \iota_{i_0}^{\beta,+}(s, t), v_{i_0}(s_v, t)) < \delta_0, \quad (5.37)$$

then $u' = u_T$ for some $T > T_0$.

Proof. See Section 5.6. □

Proof of Theorem 5.3.8. Abbreviate

$$\Phi^{\beta\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha) \rightarrow \text{CF}_*(L_0^\beta, L_1^\beta)$$

for the Floer chain map associated to the Floer data (j, H, J) via (5.28). Fix two tuples x^α, z^β of critical points and consider the 1-dimensional manifold $\mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$. Suppose y^β is another tuple of critical points and

$$u \in \mathcal{M}^0(x^\alpha, y^\beta; j, H, J), \quad v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta).$$

are solutions of the relevant Floer equation as in Theorem 5.4.3. Then the image of the gluing map $T \mapsto u_T$ in Theorem 5.4.3 is an end of the 1-manifold $\mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$. An analogous result shows that every pair of solutions $v \in \mathcal{M}^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha)$ and $u \in \mathcal{M}^0(y^\alpha, z^\beta; j, H, J)$ also determines an end of the 1-manifold $\mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$. Combining Theorem 5.4.3 with the standard compactness theorem in Floer theory and our transversality assumptions, we find that every sequence in $\mathcal{M}_\varepsilon^1(x^\alpha, z^\beta; j, H, J)$, that does not have a convergent subsequence, must be contained (after eliminating finitely elements of the sequence) in the union of the images of these gluing maps. This shows that the 1-dimensional manifold $\mathcal{M}_\varepsilon^1(x^\alpha, z^\beta; j, H, J)$ admits a compactification $\overline{\mathcal{M}}_\varepsilon^1(x^\alpha, z^\beta; j, H, J)$, which is a compact 1-manifold with boundary

$$\begin{aligned} \partial \overline{\mathcal{M}}_\varepsilon^1(x^\alpha, z^\beta; j, H, J) &= \bigcup_{y^\alpha} \bigcup_{\delta} \widehat{\mathcal{M}}_\delta^1(x^\alpha, y^\beta; H^\alpha, J^\alpha) \times \mathcal{M}_{\varepsilon-\delta}^0(y^\beta, z^\beta; j, H, J) \\ &\quad \cup \bigcup_{y^\beta} \bigcup_{\delta} \mathcal{M}_\delta^0(x^\alpha, y^\beta; j, H, J) \times \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y^\beta, z^\beta; H^\beta, J^\beta). \end{aligned}$$

Since every compact 1-manifold has an even number of boundary points, this implies

$$\begin{aligned} &\sum_{y^\alpha} \sum_{\delta} \# \widehat{\mathcal{M}}_\delta^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha) \cdot \# \mathcal{M}_{\varepsilon-\delta}^0(y^\alpha, z^\beta; j, H, J) \\ &- \sum_{y^\beta} \sum_{\delta} \# \mathcal{M}_\delta^0(x^\alpha, y^\beta; j, H, J) \cdot \# \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y^\beta, z^\beta; H^\beta, J^\beta) \\ &\in 2\mathbb{Z} \end{aligned}$$

for all ε and all x^α, z^β . This is equivalent to the formula

$$\partial^\beta \circ \Phi^{\beta\alpha} = \Phi^{\beta\alpha} \circ \partial^\alpha$$

and proves Theorem 5.3.8. \square

Chain Homotopy Equivalence

In this section we prove Theorem 5.3.10.

5.4.4. Let

$$\mathcal{L}^\alpha = (L_0^\alpha, L_1^\alpha, H^\alpha, J^\alpha), \quad \mathcal{L}^\beta = (L_0^\beta, L_1^\beta, H^\beta, J^\beta)$$

be as in 5.4.2 and let (Σ, L) be a string cobordism from (L_0^α, L_1^α) to (L_0^β, L_1^β) . Choose two regular sets of Floer data (j_0, H_0, J_0) and (j_1, H_1, J_1) on Σ from (H^α, J^α) to (H^β, J^β) , and let $\{j_\lambda, H_\lambda, J_\lambda\}_{0 \leq \lambda \leq 1}$ be a regular homotopy of Floer data from (j_0, H_0, J_0) to (j_1, H_1, J_1) as in 5.3.9.

Theorem 5.4.5 (Floer Gluing/Chain Homotopy Equivalence). *Let (Σ, L) and $(j_\lambda, H_\lambda, J_\lambda)$ be as in 5.4.4. Fix three tuples*

$$x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}, \quad y^\beta = \{y_i^\beta\}_{i \in I^\beta}, \quad z^\beta = \{z_i^\beta\}_{i \in I^\beta}$$

with $x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$ and $y_i^\beta, z_i^\beta \in \mathcal{C}(L_{i0}^\beta, L_{i1}^\beta; H_i^\beta)$. Let

$$0 < \lambda < 1, \quad u \in \mathcal{M}^{-1}(x^\alpha, y^\beta; j_\lambda, H_\lambda, J_\lambda), \quad v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta),$$

such that $v_i(s, t) = y_i^\beta(t) = z_i^\beta(t)$ for $i \neq i_0$ and $\mu_H(v_{i_0}) = 1$. Fix any nonempty open set $W_0 \subset \Sigma \setminus \text{im } \iota_{i_0}^{\beta,+}$ and a real number $s_v \in \mathbb{R}$. Then there exist constants $T_0 > 0$ and $\delta_0 > 0$ and a smooth map

$$(T_0, \infty) \rightarrow \mathcal{M}^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}) : T \mapsto (\lambda_T, u_T) \quad (5.38)$$

satisfying the following conditions.

- (i) *The map (5.38) is a diffeomorphism onto its image.*
- (ii) *u_T converges to u , the maps $(s, t) \mapsto u_T \circ \iota_{i_0}^{\beta,+}(s + T, t)$ converge to v_{i_0} , and λ_T converges to λ as T tends to infinity; in the first two cases the convergence is uniform with all derivatives on every compact subset of Σ , respectively $\mathbb{R} \times [0, 1]$. Moreover,*

$$\lim_{T \rightarrow \infty} \left(\sup_{\Sigma \setminus \text{im } \iota_{i_0}^{\beta,+}} d(u_T, u) + \sup_{s \geq 0} \sup_{0 \leq t \leq 1} d(u_T \circ \iota_{i_0}^{\beta,+}(s + T, t), v_{i_0}(s, t)) \right) = 0.$$

- (iii) *$\mathcal{A}_H(u_T) = \mathcal{A}_H(u) + \mathcal{A}_H(v_{i_0})$ for every $T > T_0$.*

- (iv) *If $0 < \lambda' < 1$ and $u' \in \mathcal{M}^0(x^\alpha, z^\beta; j_{\lambda'}, H_{\lambda'}, J_{\lambda'})$ satisfies*

$$\begin{aligned} |\lambda' - \lambda| &< \delta_0, & \mathcal{A}_H(u') &= \mathcal{A}_H(u) + \mathcal{A}_H(v_{i_0}), \\ \sup_{W_0} d(u', u) &< \delta_0, & \inf_{s \geq 0} \sup_{0 \leq t \leq 1} d(u' \circ \iota_{i_0}^{\beta,+}(s, t), v_{i_0}(s_v, t)) &< \delta_0, \end{aligned} \quad (5.39)$$

then $(\lambda', u') = (\lambda_T, u_T)$ for some $T > T_0$.

Proof. See Section 5.6. □

Proof of Theorem 5.3.10. Fix two tuples x^α, z^β of critical points and consider the 1-dimensional manifold $\mathcal{M}^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$. Suppose y^β is another tuple of critical points and

$$0 < \lambda < 1, \quad u \in \mathcal{M}^{-1}(x^\alpha, y^\beta; j_\lambda, H_\lambda, J_\lambda), \quad v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta).$$

are solutions of the relevant Floer equation as in Theorem 5.4.5. Then the image of the gluing map $T \mapsto u_T$ in Theorem 5.4.5 is an end of the 1-manifold $\mathcal{M}^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$. In a similar way, every pair of solutions $v \in \mathcal{M}^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha)$ and $u \in \mathcal{M}^{-1}(y^\alpha, z^\beta; j_\lambda, H_\lambda, J_\lambda)$ with $0 < \lambda < 1$ also determines an end of the 1-manifold $\mathcal{M}^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$. Combining Theorem 5.4.5 with the standard compactness theorem in Floer theory and our transversality assumptions, we find that every sequence in $\mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$ that does not have a convergent subsequence, is contained (after eliminating finitely elements of the sequence) in the union of the images of these gluing maps. This shows that the 1-dimensional manifold $\mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$ admits a compactification which is a compact 1-manifold with boundary, denoted by $\overline{\mathcal{M}}_\varepsilon^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda)$, whose boundary is given by

$$\begin{aligned} \partial \overline{\mathcal{M}}_\varepsilon^0(x^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) \\ = \mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; j_0, H_0, J_0) \cup \mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; j_1, H_1, J_1) \\ \cup \bigcup_{y^\alpha} \bigcup_{\delta} \widehat{\mathcal{M}}_\delta^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha) \times \mathcal{M}_{\varepsilon-\delta}^{-1}(y^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) \\ \cup \bigcup_{y^\beta} \bigcup_{\delta} \mathcal{M}_\delta^{-1}(x^\alpha, y^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) \times \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y^\beta, z^\beta; H^\beta, J^\beta). \end{aligned}$$

Since every compact 1-manifold has an even number of boundary points, this implies

$$\begin{aligned} & \# \mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; j_1, H_1, J_1) - \# \mathcal{M}_\varepsilon^0(x^\alpha, z^\beta; j_0, H_0, J_0) \\ & - \sum_{y^\alpha} \sum_{\delta} \# \widehat{\mathcal{M}}_\delta^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha) \cdot \# \mathcal{M}_{\varepsilon-\delta}^{-1}(y^\alpha, z^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) \\ & - \sum_{y^\beta} \sum_{\delta} \# \mathcal{M}_\delta^{-1}(x^\alpha, y^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) \cdot \# \widehat{\mathcal{M}}_{\varepsilon-\delta}^1(y^\beta, z^\beta; H^\beta, J^\beta) \\ & \in 2\mathbb{Z} \end{aligned}$$

for all ε and all x^α, z^β . This is equivalent to the formula

$$\Phi_1^{\beta\alpha} - \Phi_0^{\beta\alpha} = \partial^\beta \circ \Psi^{\beta\alpha} + \Psi^{\beta\alpha} \circ \partial^\alpha$$

and proves Theorem 5.3.10. □

Catenation

In this section we prove Theorem 5.3.12.

5.4.6. Let (M, ω) be a compact symplectic manifold. Fix three objects

$$(L_0^\nu, L_1^\nu) = \{L_{i0}^\nu, L_{i1}^\nu\}_{i \in I^\nu}, \quad \nu = \alpha, \beta, \gamma,$$

in $\mathcal{L}(M, \omega)$ and three collections

$$(H^\nu, J^\nu) = \{H_i^\nu, J_i^\nu\}_{i \in I^\nu}, \quad \nu = \alpha, \beta, \gamma,$$

such that (H_i^ν, J_i^ν) is a regular pair for (L_{i0}^ν, L_{i1}^ν) when $i \in I^\nu$. Choose regular framed string cobordisms

$$\mathcal{S}^{\alpha\beta} = (\Sigma^{\alpha\beta}, L^{\alpha\beta}, j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$$

from $(L_0^\alpha, L_1^\alpha, H^\alpha, J^\alpha)$ to $(L_0^\beta, L_1^\beta, H^\beta, J^\beta)$ and

$$\mathcal{S}^{\beta\gamma} = (\Sigma^{\beta\gamma}, L^{\beta\gamma}, j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$$

from $(L_0^\beta, L_1^\beta, H^\beta, J^\beta)$ to $(L_0^\gamma, L_1^\gamma, H^\gamma, J^\gamma)$ (see 5.3.3 and 5.3.6). For $T > 0$ denote by $(\Sigma_T^{\alpha\gamma}, L_T^{\alpha\gamma})$ the T -catenation of the string cobordisms $(\Sigma^{\alpha\beta}, L^{\alpha\beta})$ and $(\Sigma^{\beta\gamma}, L^{\beta\gamma})$ (see Definition 5.3.1). This catenation is equipped with Floer data

$$(j_T^{\alpha\gamma}, H_T^{\alpha\gamma}, J_T^{\alpha\gamma}),$$

defined by restricting the Floer data in $(j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$ to $(\Sigma_{2T}^{\alpha\beta}, L_{2T}^{\alpha\beta})$ and restricting the Floer data $(j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$ to $(\Sigma_{2T}^{\beta\gamma}, L_{2T}^{\beta\gamma})$ (see equation (5.21)).

5.4.7. Next we formulate a gluing theorem that holds uniformly for Hamiltonian perturbations in neighborhoods of $H^{\alpha\beta}$ and $H^{\beta\gamma}$. Thus we denote by $\mathcal{H}^{\alpha\beta}$ the space of smooth 1-forms

$$h^{\alpha\beta} : T\Sigma^{\alpha\beta} \rightarrow \Omega^0(M)$$

with support in the compact set

$$\Sigma^{\alpha\beta} \setminus \left(\bigcup_{i \in I^\alpha} \text{im } \iota_i^{\alpha,-} \cup \bigcup_{i \in I^\beta} \text{im } \iota_i^{\beta,+} \right).$$

Likewise, we denote by $\mathcal{H}^{\beta\gamma}$ the space of smooth 1-forms

$$h^{\beta\gamma} : T\Sigma^{\beta\gamma} \rightarrow \Omega^0(M)$$

with support in the compact set

$$\Sigma^{\beta\gamma} \setminus \left(\bigcup_{i \in I^\beta} \text{im } \iota_i^{\beta,-} \cup \bigcup_{i \in I^\gamma} \text{im } \iota_i^{\gamma,+} \right).$$

Theorem 5.4.8 (Floer Gluing/Catenation). *Let $(\Sigma^{\alpha\beta}, L^{\alpha\beta})$, $(\Sigma^{\beta\gamma}, L^{\beta\gamma})$, and $(j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$, $(j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$ be as in 5.4.6. Fix three tuples*

$$x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}, \quad x^\beta = \{x_i^\beta\}_{i \in I^\beta}, \quad x^\gamma = \{x_i^\gamma\}_{i \in I^\gamma}$$

with $x_i^\nu \in \mathcal{C}(L_{i0}^\nu, L_{i1}^\nu; H_i^\nu)$ for $i \in I^\nu$ and $\nu = \alpha, \beta, \gamma$, and let

$$u^{\alpha\beta} \in \mathcal{M}^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta}), \quad u^{\beta\gamma} \in \mathcal{M}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma}).$$

Fix two nonempty open sets

$$W^{\alpha\beta} \subset \Sigma^{\alpha\beta} \setminus \text{im } \iota^{\beta,+}, \quad W^{\beta\gamma} \subset \Sigma^{\beta\gamma} \setminus \text{im } \iota^{\beta,-},$$

where $\text{im } \iota^{\beta,\pm} := \bigcup_{i \in I^\beta} \text{im } \iota_i^{\beta,\pm}$. Then there exist a convex open neighborhood $\mathcal{H}_0 \subset \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$ of the origin, constants $T_0 > 0$ and $\delta_0 > 0$, smooth families

$$\begin{aligned} u_h^{\alpha\beta} &\in \mathcal{M}^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta} + h^{\alpha\beta}, J^{\alpha\beta}), \\ u_h^{\beta\gamma} &\in \mathcal{M}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma} + h^{\beta\gamma}, J^{\beta\gamma}), \end{aligned}$$

(parametrized by $h = (h^{\alpha\beta}, h^{\beta\gamma}) \in \mathcal{H}_0$), and a smooth family

$$u_{h,T} \in \mathcal{M}^0(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H + h)_T^{\alpha\gamma}, J_T^{\alpha\gamma}), \quad h \in \mathcal{H}_0, \quad T > T_0, \quad (5.40)$$

satisfying the following conditions.

- (i) For $h \in \mathcal{H}_0$ and $T \geq T_0$ the solutions $u_h^{\alpha\beta} : \Sigma^{\alpha\beta} \rightarrow M$, $u_h^{\beta\gamma} : \Sigma^{\beta\gamma} \rightarrow M$ and $u_{h,T} : \Sigma_T^{\alpha\gamma} \rightarrow M$ of the Floer equation are regular in the sense that the linearized operators are bijective. Moreover, $u_0^{\alpha\beta} = u^{\alpha\beta}$ and $u_0^{\beta\gamma} = u^{\beta\gamma}$.
- (ii) For every $h \in \mathcal{H}_0$ the maps $u_{h,T}^{\alpha\gamma}$ converge to $u_h^{\alpha\beta}$, uniformly with all derivatives on every compact subset of $\Sigma^{\alpha\beta}$, and they converge to $u_h^{\beta\gamma}$, uniformly with all derivatives on every compact subset of $\Sigma^{\beta\gamma}$ (as T tends to infinity). Moreover,

$$\lim_{T \rightarrow \infty} \sup_{h \in \mathcal{H}_0} \left(\sup_{\Sigma^{\alpha\beta} \setminus \text{im } \iota^{\beta,+}} d(u_{h,T}^{\alpha\gamma}, u_h^{\alpha\beta}) + \sup_{\Sigma^{\beta\gamma} \setminus \text{im } \iota^{\beta,-}} d(u_{h,T}^{\alpha\gamma}, u_h^{\beta\gamma}) \right) = 0.$$

- (iii) For $h \in \mathcal{H}_0$ and $T \geq T_0$ we have

$$\mathcal{A}_H(u_{h,T}^{\alpha\gamma}) = \mathcal{A}_H(u_h^{\alpha\beta}) + \mathcal{A}_H(u_h^{\beta\gamma}).$$

- (iv) If $h \in \mathcal{H}_0$, $T \geq T_0$, and $u' \in \mathcal{M}^0(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H + h)_T^{\alpha\gamma}, J_T^{\alpha\gamma})$ satisfy

$$\begin{aligned} \mathcal{A}_H(u') &= \mathcal{A}_H(u_h^{\alpha\beta}) + \mathcal{A}_H(u_h^{\beta\gamma}), \\ \sup_{W^{\alpha\beta}} d(u', u_h^{\alpha\beta}) &< \delta_0, \quad \sup_{W^{\beta\gamma}} d(u', u_h^{\beta\gamma}) < \delta_0, \end{aligned} \quad (5.41)$$

then $u' = u_{h,T}^{\alpha\gamma}$.

Proof. See Section 5.6. □

Proof of Theorem 5.3.12. On the chain level the composition

$$\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\gamma, L_1^\gamma; H^\gamma)$$

is given by

$$\Phi^{\gamma\beta} \Phi^{\beta\alpha} x^\alpha = \sum_{x^\gamma} \sum_{\varepsilon} n_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) e^{-\varepsilon} x^\gamma,$$

where the number $n_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) \in \mathbb{Z}/2\mathbb{Z}$ is defined by

$$\begin{aligned} n_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) &:= \sum_{\delta} \sum_{x^\beta} n_\delta^{\alpha\beta}(x^\alpha, x^\beta) n_{\varepsilon-\delta}^{\beta\gamma}(x^\beta, x^\gamma), \\ n_\delta^{\alpha\beta}(x^\alpha, x^\beta) &:= \#_2 \mathcal{M}_\delta^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta}), \\ n_{\varepsilon-\delta}^{\beta\gamma}(x^\beta, x^\gamma) &:= \#_2 \mathcal{M}_{\varepsilon-\delta}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma}). \end{aligned} \quad (5.42)$$

We prove in four steps that $\Phi^{\gamma\beta} \circ \Phi^{\beta\alpha}$ is chain homotopy equivalent to the Floer chain map $\Phi^{\gamma\alpha}$ associated to regular Floer data on the catenation of the string cobordisms $\Sigma^{\alpha\beta}$ and $\Sigma^{\beta\gamma}$.

Step 1. Fix a sequence of real numbers $c_0 < c_1 < c_2 < \dots$ diverging to ∞ . Then there exist sequences of real numbers $T_\nu > 0$ and $\delta_\nu > 0$ and a sequence of convex open neighborhoods $\mathcal{H}_\nu \subset \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$ of the origin satisfying the following conditions.

- (a) For every $\nu \in \mathbb{N}_0$ we have $T_\nu < T_{\nu+1}$, $\delta_{\nu+1} < \delta_\nu$, and $\mathcal{H}_{\nu+1} \subset \mathcal{H}_\nu$. Moreover, T_ν diverges to infinity.
- (b) The assertions of Theorem 5.4.8 hold with $\mathcal{H}_0, T_0, \delta_0$ replaced by $\mathcal{H}_\nu, T_\nu, \delta_\nu$ for all $x^\alpha, x^\beta, x^\gamma$ and all $u^{\alpha\beta}$ and $u^{\beta\gamma}$ such that

$$\mathcal{A}_H(u^{\alpha\beta}) + \mathcal{A}_H(u^{\beta\gamma}) \leq c_\nu.$$

- (c) For all x^α, x^γ , all $\varepsilon \leq c_\nu$, all $h \in \mathcal{H}_\nu$, and all $T \geq T_\nu$ the map

$$\begin{aligned} \bigcup_{\delta} \bigcup_{x^\beta} \mathcal{M}_\delta^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta}) \times \mathcal{M}_{\varepsilon-\delta}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma}) \\ \rightarrow \mathcal{M}_\varepsilon^0(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H+h)_T^{\alpha\gamma}, J_T^{\alpha\gamma}) : (u^{\alpha\beta}, u^{\beta\gamma}) \mapsto u_{h,T}^{\alpha\gamma} \end{aligned} \quad (5.43)$$

of Theorem 5.4.8 is bijective.

- (d) For all x^α, x^γ , all $\varepsilon \leq c_\nu$, all $h \in \mathcal{H}_\nu$, and all $T \geq T_\nu$ we have

$$\mathcal{M}_\varepsilon^{-1}(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H+h)_T^{\alpha\gamma}, J_T^{\alpha\gamma}) = \emptyset.$$

The proof is by induction on ν . For $\nu = 0$ it follows from Theorem 5.4.8 that $\mathcal{H}_0, T_0, \delta_0$ can be chosen such that (b) holds (because there are only finitely

many pairs $(u^{\alpha\beta}, u^{\beta\gamma})$ as in Theorem 5.4.8 with $\mathcal{A}_H(u^{\alpha\beta}) + \mathcal{A}_H(u^{\beta\gamma}) \leq c_0$. After shrinking \mathcal{H}_0 and increasing T_0 , if necessary, assertion (b) continues to be valid and we claim that (c) and (d) hold as well.

We prove this first for (d). Suppose otherwise. Then there exist critical points x^α, x^γ , a sequence of Hamiltonian perturbations $h_k \in \mathcal{H}_0$, and a sequence of real numbers $T_k > T_0$, such that h_k converges to zero in the C^∞ -topology, T_k diverges to ∞ , and for every k

$$\bigcup_{\varepsilon \leq c_0} \mathcal{M}_\varepsilon^{-1}(x^\alpha, x^\gamma; j_{T_k}^{\alpha\gamma}, (H + h_k)_{T_k}^{\alpha\gamma}, J_{T_k}^{\alpha\gamma}) \neq \emptyset.$$

Choose a sequence of pairs (ε_k, u_k) such that

$$u_k^{\alpha\gamma} \in \mathcal{M}_{\varepsilon_k}^{-1}(x^\alpha, x^\gamma; j_{T_k}^{\alpha\gamma}, (H + h_k)_{T_k}^{\alpha\gamma}, J_{T_k}^{\alpha\gamma}), \quad \varepsilon_k \leq c_0.$$

Then the standard Floer–Gromov compactness theorem asserts that a suitable subsequence of $u_k^{\alpha\gamma}$ converges, modulo bubbling, to a catenation of finitely many Floer trajectories for (H^α, J^α) running from x^α to some critical point y^α , an element of $\mathcal{M}(y^\alpha, y^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$ for some critical point y^β , finitely many Floer trajectories for (H^β, J^β) running from y^β to some critical point z^β , an element of $\mathcal{M}(z^\beta, z^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$ for some critical point z^γ , and finitely many Floer trajectories for (H^γ, J^γ) running from z^γ to x^γ . By monotonicity, the total Fredholm index of this catenation must be less than or equal to minus one. On the other hand, by our transversality hypotheses, the total Fredholm index of this catenation must be bigger than or equal to zero. This contradiction shows that (d) holds after shrinking \mathcal{H}_0 and increasing T_0 , if necessary.

Next we prove that (c) holds for a suitable pair \mathcal{H}_0, T_0 . It follows directly from Theorem 5.4.8 (ii) that the map (5.43) is injective for h sufficiently close to zero and T sufficiently large. Assume, by contradiction, that there is a pair of critical points x^α, x^γ , a sequence of Hamiltonian perturbations $h_k \in \mathcal{H}_0$ converging to zero in the C^∞ topology, a sequence $T_k \rightarrow \infty$, a sequence $\varepsilon_k \leq c_0$, and a sequence

$$u_k^{\alpha\gamma} \in \mathcal{M}_{\varepsilon_k}^0(x^\alpha, x^\gamma; j_{T_k}^{\alpha\gamma}, (H + h_k)_{T_k}^{\alpha\gamma}, J_{T_k}^{\alpha\gamma})$$

that does not belong to the image of the map (5.43) for $(h, T) = (h_k, T_k)$. Then it follows from monotonicity, transversality, and the same compactness argument as in the proof of (d) that a suitable subsequence of $u_k^{\alpha\gamma}$ converges to a catenation of an element $u^{\alpha\beta} \in \mathcal{M}(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$ and an element $u^{\beta\gamma} \in \mathcal{M}(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$ for some δ and some x^β . Moreover there cannot be any bubbling. This means that, for k sufficiently large, the map

$u_k^{\alpha\gamma}$ satisfies the assumptions of Theorem 5.4.8 (iv) and hence does after all belong to the image of the map (5.43), a contradiction. Thus we have proved Step 1 for $\nu = 0$.

Now let $\nu \geq 1$ and suppose that $\mathcal{H}_{\nu-1}, T_{\nu-1}, \delta_{\nu-1}$ have been constructed. Using Theorem 5.4.8 and the above compactness argument, we find a convex open neighborhood $\mathcal{H}_\nu \subset \mathcal{H}_{\nu-1}$ of zero and real numbers $T_\nu > \max\{\nu, T_{\nu-1}\}$, $0 < \delta_\nu < \min\{1/\nu, \delta_{\nu-1}\}$ such that (b), (c), and (d) hold. This completes the induction argument and the sequences satisfy (a) by construction. Thus we have proved Step 1.

Step 2. Let $c_\nu, T_\nu, \delta_\nu, \mathcal{H}_\nu$ be as in Step 1. Choose $h_\nu \in \mathcal{H}_\nu$ such that the triple $(j_{T_\nu}^{\alpha\gamma}, (H + h_\nu)_{T_\nu}^{\alpha\gamma}, J_{T_\nu}^{\alpha\gamma})$ is regular and denote by

$$\Phi_\nu^{\gamma\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\gamma, L_1^\gamma; H^\gamma)$$

the associated homomorphism on the Floer chain complex. Then the coefficients of $\Phi_\nu^{\gamma\alpha}$ agree with those of $\Phi^{\gamma\beta}\Phi^{\beta\alpha}$ for action values $\mathcal{A}_{H+h_\nu}(u) \leq c_\nu$.

The operator $\Phi_\nu^{\alpha,\gamma}$ is given by

$$\Phi_\nu^{\gamma\alpha} x^\alpha = \sum_{x^\gamma} \sum_{\varepsilon} n_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) e^{-\varepsilon} x^\gamma,$$

where the number

$$n_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) := \#_2 \mathcal{M}_\varepsilon^0(x^\alpha, x^\gamma; j_{T_\nu}^{\alpha\gamma}, (H + h_\nu)_{T_\nu}^{\alpha\gamma}, J_{T_\nu}^{\alpha\gamma}).$$

Since $h_\nu \in \mathcal{H}_\nu$, it follows from condition (c) in Step 1 that

$$\begin{aligned} n_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) &= \sum_{\delta} \sum_{x^\beta} n_\delta^{\alpha\beta}(x^\alpha, x^\beta) n_{\varepsilon-\delta}^{\beta\gamma}(x^\beta, x^\gamma) \\ &= n_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) \end{aligned}$$

for $\varepsilon \leq c_\nu$. (For the last step see equation (5.42).) This proves Step 2.

Step 3. Let $c_\nu, T_\nu, \delta_\nu, \mathcal{H}_\nu$ be as in Step 1 and let $h_\nu, \Phi_\nu^{\gamma\alpha}$ be as in Step 2. Then there exists a sequence of chain homotopy equivalences $\Psi_\nu^{\gamma\alpha}$ so that

$$\Phi_{\nu+1}^{\gamma\alpha} - \Phi_\nu^{\gamma\alpha} = \partial^\gamma \circ \Psi_\nu^{\gamma\alpha} + \Psi_\nu^{\gamma\alpha} \circ \partial^\alpha$$

and all coefficients of $\Psi_\nu^{\gamma\alpha}$ have the form $\lambda = \sum_{\varepsilon > c_\nu} \lambda_\varepsilon e^{-\varepsilon}$.

Fix a string cobordism $(\Sigma^{\alpha\gamma}, L^{\alpha\gamma})$ from (L_0^α, L_1^α) to (L_0^γ, L_1^γ) that is equivalent to the catenation $(\Sigma_T^{\alpha\gamma}, L_T^{\alpha\gamma}) = (\Sigma^{\alpha\gamma}, L^{\alpha\beta}) \#_T (\Sigma^{\beta\gamma}, L^{\beta\gamma})$ in Definition 5.3.1 for all $T > 0$. Choose a family of diffeomorphisms

$$\phi_T : \Sigma^{\alpha\gamma} \rightarrow \Sigma_T^{\alpha\gamma} = \Sigma^{\alpha\beta} \#_T \Sigma^{\beta\gamma}, \quad T \geq 1,$$

as follows. Our fixed string cobordism is $\Sigma^{\alpha\gamma} := \Sigma_1^{\alpha\gamma}$ associated to $T = 1$. Now choose a smooth function $\rho : \mathbb{R} \rightarrow [-1, 1]$ such that $\rho' \geq 0$ and

$$\rho(s) = \begin{cases} -1, & \text{for } s \leq -1/2, \\ 1, & \text{for } s \geq 1/2. \end{cases}$$

For $T \geq 1$ define $\rho_T : [-1, 1] \rightarrow [-T, T]$ by

$$\rho_T(s) := s + (T - 1)\rho(s)$$

Then $\rho_T : [-1, 1] \rightarrow [-T, T]$ is a diffeomorphism with $\rho_T(\pm 1) = \pm T$ and $\rho'_T(s) = 1$ for $|s| \geq 1/2$. Now define ϕ_T by

$$\begin{aligned} \phi_T(\iota_i^{\beta,+}(s+1, t)) &:= \iota_i^{\beta,+}(\rho_T(s) + T, t), \\ \phi_T(\iota_i^{\beta,-}(s-1, t)) &:= \iota_i^{\beta,-}(\rho_T(s) - T, t) \end{aligned}$$

for $|s| \leq 1$, and by the identity on $\Sigma^{\alpha\beta} \setminus \bigcup_{i \in I^\beta} \text{im} \iota_i^{\beta,+}$ and $\Sigma^{\beta\gamma} \setminus \bigcup_{i \in I^\beta} \text{im} \iota_i^{\beta,-}$. Now it follows from standard transversality arguments in Floer theory that there exists a smooth homotopy $\{h_T\}_{T_\nu \leq T \leq T_{\nu+1}}$ in \mathcal{H}_ν from h_ν to $h_{\nu+1}$ such that the pullbacks $\phi_T^*(j_T^{\alpha\gamma}, (H + h_T)_T^{\alpha\gamma}, J_T^{\alpha\gamma})$ of the resulting Floer data on $\Sigma_T^{\alpha\gamma}$ to $\Sigma^{\alpha\gamma}$ under ϕ_T define a regular homotopy of Floer data as in 5.3.9. Denote by $\Psi_\nu^{\gamma\alpha} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\gamma, L_1^\gamma; H^\gamma)$ the homomorphism associated to this homotopy via (5.31). Then, by Theorem 5.3.10, we have

$$\Phi_{\nu+1}^{\gamma\alpha} - \Phi_\nu^{\gamma\alpha} = \partial^\gamma \circ \Psi_\nu^{\gamma\alpha} + \Psi_\nu^{\gamma\alpha} \circ \partial^\alpha.$$

By equation (5.31) the operator $\Psi_\nu^{\gamma\alpha}$ is given by

$$\Psi_\nu^{\gamma\alpha} x^\alpha = \sum_{x^\beta} \sum_{\varepsilon} N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) e^{-\varepsilon} x^\gamma,$$

where

$$N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) := \sum_{T_\nu \leq T \leq T_{\nu+1}} \#_2 \mathcal{M}_\varepsilon^{-1}(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H + h_T)_T^{\alpha\gamma}, J_T^{\alpha\gamma}). \quad (5.44)$$

Since $h_T \in \mathcal{H}_\nu$ and $T \geq T_\nu$ it follows from (d) in Step 1 that the moduli space $\mathcal{M}_\varepsilon^{-1}(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H + h_T)_T^{\alpha\gamma}, J_T^{\alpha\gamma})$ is empty, and hence $N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) = 0$, whenever $\varepsilon \leq c_\nu$. This proves Step 3.

Step 4. Let $c_\nu, T_\nu, \delta_\nu, \mathcal{H}_\nu$ be as in Step 1, let $h_\nu, \Phi_\nu^{\gamma\alpha}$ be as in Step 2, and let $\Psi_\nu^{\gamma\alpha}$ be as in Step 3. Then the infinite sum

$$\Psi^{\gamma\alpha} := \Psi_0^{\gamma\alpha} + \Psi_1^{\gamma\alpha} + \Psi_2^{\gamma\alpha} + \cdots$$

defines a homomorphism from $\text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha)$ to $\text{CF}_*(L_0^\gamma, L_1^\gamma; H^\gamma)$ and

$$\Phi^{\gamma\beta}\Phi^{\beta\alpha} - \Phi_0^{\gamma\alpha} = \partial^\gamma \circ \Psi^{\gamma\alpha} + \Psi^{\gamma\alpha} \circ \partial^\alpha.$$

For x^α , x^γ , and ε define

$$N_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) := \sum_{\nu=0}^{\infty} N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma),$$

where the numbers $N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma)$ are given by (5.44). This sum is finite and $\sum_\varepsilon N_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma)e^{-\varepsilon} \in \Lambda$ because $N_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) = 0$ whenever $c_\nu \geq c \geq \varepsilon$. Hence the infinite sum

$$\Psi^{\alpha\gamma} := \sum_{\nu=0}^{\infty} \Psi_\nu^{\alpha\gamma} : \text{CF}_*(L_0^\alpha, L_1^\alpha; H^\alpha) \rightarrow \text{CF}_*(L_0^\gamma, L_1^\gamma; H^\gamma)$$

is well defined and given by

$$\Psi^{\gamma\alpha}x^\alpha = \sum_{x^\gamma} \sum_{\varepsilon} N_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma)e^{-\varepsilon}x^\gamma.$$

Denote

$$n_\varepsilon^\alpha(x^\alpha, y^\alpha) := \#_2 \widehat{\mathcal{M}}_\varepsilon^1(x^\alpha, y^\alpha; H^\alpha, J^\alpha)$$

and similarly for β and γ . Then

$$\begin{aligned} n_\varepsilon^{\alpha\gamma}(x^\alpha, x^\gamma) - n_{0,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) &= \sum_{\nu=0}^{\infty} (n_{\nu+1,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma) - n_{\nu,\varepsilon}^{\alpha\gamma}(x^\alpha, x^\gamma)) \\ &= \sum_{\nu=0}^{\infty} \sum_{y^\gamma} \sum_{\delta} N_{\nu,\delta}^{\alpha\gamma}(x^\alpha, y^\gamma) n_{\varepsilon-\delta}^\gamma(y^\gamma, x^\gamma) \\ &\quad + \sum_{\nu=0}^{\infty} \sum_{y^\alpha} \sum_{\delta} n_\delta^\alpha(x^\alpha, y^\alpha) N_{\nu,\varepsilon-\delta}^{\alpha\gamma}(y^\alpha, x^\gamma) \\ &= \sum_{y^\gamma} \sum_{\delta} N_\delta^{\alpha\gamma}(x^\alpha, y^\gamma) n_{\varepsilon-\delta}^\gamma(y^\gamma, x^\gamma) \\ &\quad + \sum_{y^\alpha} \sum_{\delta} n_\delta^\alpha(x^\alpha, y^\alpha) N_{\varepsilon-\delta}^{\alpha\gamma}(y^\alpha, x^\gamma). \end{aligned}$$

Here the finiteness of the first sum on the right follows from Step 2 and the finiteness of the next two sums follows from Step 3. This proves Step 4 and Theorem 5.3.12. \square

5.5 Truncated String Cobordism

In this section we carry over the results of Theorem 4.1.5 to the case of truncated surfaces. Fix two objects

$$(L_0^\alpha, L_1^\alpha) = \{L_{i0}^\alpha, L_{i1}^\alpha\}_{i \in I^\alpha}, \quad (L_0^\beta, L_1^\beta) = \{L_{i0}^\beta, L_{i1}^\beta\}_{i \in I^\beta}$$

in $\mathcal{L}(M, \omega)$ and two collections

$$(H^\alpha, J^\alpha) = \{H_i^\alpha, J_i^\alpha\}_{i \in I^\alpha}, \quad (H^\beta, J^\beta) = \{H_i^\beta, J_i^\beta\}_{i \in I^\beta}$$

such that (H_i^α, J_i^α) is a regular pair for $(L_{i0}^\alpha, L_{i1}^\alpha)$ when $i \in I^\alpha$ and (H_i^β, J_i^β) is a regular pair for $(L_{i0}^\beta, L_{i1}^\beta)$ when $i \in I^\beta$. Let

$$(\Sigma, L) = \left(\Sigma, \{L_z\}_{z \in \partial \Sigma}, \{\iota_i^{\alpha, -}\}_{i \in I^\alpha}, \{\iota_i^{\beta, +}\}_{i \in I^\beta} \right)$$

be a string cobordism from (L_0^α, L_1^α) to (L_0^β, L_1^β) and let (j, H, J) be a regular set of Floer data on Σ from (H^α, J^α) to (H^β, J^β) . Fix two tuples $x^\alpha = \{x_i^\alpha\}_{i \in I^\alpha}$ and $y^\beta = \{y_i^\beta\}_{i \in I^\beta \setminus \{i_0\}}$ with $x_i^\alpha \in \mathcal{C}(L_{i0}^\alpha, L_{i1}^\alpha; H_i^\alpha)$ and $y_i^\beta \in \mathcal{C}(L_{i0}^\beta, L_{i1}^\beta; H_i^\beta)$. Denote with Σ_0 a truncated surface

$$\Sigma_0 := \Sigma \setminus \iota_{i_0}^{\beta, +}((s_0, \infty) \times [0, 1]).$$

We define the moduli space of perturbed holomorphic curves whose domain is truncated surface Σ_0 and we prove that it is an infinite dimensional manifold and that the restriction to the non-Lagrangian boundary is injective immersion.

$$\mathcal{M} := \left\{ u \in W_{\text{loc}}^{2,2}(\Sigma_0, M) \left| \begin{array}{l} u \text{ satisfies (5.23), } E_H(u) < \infty, \\ \lim_{s \rightarrow -\infty} u(\iota_i^{\alpha, -}(s, t)) = x_i^\alpha(t), i \in I^\alpha \\ \lim_{s \rightarrow \infty} u(\iota_i^{\beta, +}(s, t)) = y_i^\beta(t), i \in I^\beta \setminus \{i_0\} \end{array} \right. \right\}.$$

Theorem 5.5.1. *The moduli space \mathcal{M} defined above is a Hilbert manifold and the restriction map*

$$i : \mathcal{M} \rightarrow \mathcal{P}, \quad i(u) := u(\iota_{i_0}^{\beta, +}(s_0, \cdot))$$

is an injective immersion.

Proof. As in the proof of theorem 4.1.5 we consider some base manifold \mathcal{B} which consists of $W^{2,2}(\Sigma_0, M)$ maps and which satisfy the corresponding two Lagrangian boundary conditions and a Hilbert space bundle \mathcal{E} over \mathcal{B} . The

only difference is that this time the fibers consist of $(0, 1)$ forms. The set \mathcal{M} can be seen as the zero set of the sections $\bar{\partial}_{J,H} = \mathcal{S}$ of the bundle \mathcal{E} . Thus it is enough to prove that this section is transverse to zero section or equivalently that the vertical differential D_u is surjective. Here

$$D_u : W_{bc}^{2,2}(\Sigma_0, u^*TM) \rightarrow W_{bc}^{1,2}(\Sigma_0, \Lambda^{0,1} \otimes_J u^*TM)$$

$$D_u(\xi) = \frac{1}{2} \left(\nabla \xi + \nabla_\xi X_H + J \circ (\nabla \xi + \nabla_\xi X_H) \circ j + (\nabla_\xi J) \circ d_H u \circ j \right), \quad (5.45)$$

where $X_H(u) \in \Omega^1(\Sigma_0, u^*TM)$ and $d_H u(z) \in \Omega^1(\Sigma_0, u^*TM)$ is given by $d_H u(z)(\hat{z}) = du(z)(\hat{z}) + X_H(z)(\hat{z})$. Let $\eta \in W_{bc}^{1,2}(\Sigma_0, \Lambda^{0,1} \otimes_J u^*TM)$. We show that there exists $\xi \in W_{bc}^{2,2}(\Sigma_0, u^*TM)$ such that $D_u \xi = \eta$. Let U_i , $i = 1, \dots, m$ be an open cover of Σ_0 such that each U_i is diffeomorphic to one of the following

- 1) Strip of the form $(-\infty, 0) \times (0, 1)$ or $(0, 1] \times (0, 1)$.
- 2) Open disc or half disk.

Let β_i , $i = 1, \dots, m$ be a partition of unity subordinate to the cover U_i , $\text{supp}(\beta_i) \subset U_i$. We can assume w.l.o.g. that on each U_i we have local conformal coordinates $z = s + it$. In these local coordinates the $(0, 1)$ form η can be written as $\eta_i = \beta_i \eta = \eta_1^i ds + J \eta_1^i dt$, where η_1 is a vector field along u with support in U_i . In each local chart the operator D_u can be identified with the operator

$$D_i \xi := (D_u \xi) \left(\frac{\partial}{\partial s} \right) : W_{bc}^{2,2}(U_i, u^*TM) \rightarrow W_{bc}^{1,2}(U_i, u^*TM).$$

In local coordinates on U_i the form $X_H(u) = X_F(u)ds + X_G(u)dt$, where $X_F(u), X_G(u)$ are vector fields along u . Thus, the operator D_i has the form

$$D_i \xi = \nabla_s \xi + \nabla_\xi X_F + (\nabla_\xi J)(\partial_t u + X_G) + J(u)(\nabla_t \xi + \nabla_\xi X_G)$$

Analogously as in the proof of Theorem 4.1.5 we can construct an adequate trivialization Φ of $u^*TM|_{U_i}$, which transform the operator D_i to an operator of the form (3.37). In the case that U_i is a disk or half disk, we extend first the functions X_F and X_G in such a way that we obtain an operator on the strip. Thus, the same argument as in Corollary 3.3.7 proves that each D_i is surjective. Particularly, this means that for each $\eta_i = \beta_i \eta$ there exist ξ_i such that $\text{supp}(\xi_i) \subset U_i$ and $D_u \xi_i = \eta_i$. Extend ξ_i by zero to entire Σ_0 . We have

$$D_u \left(\sum_i \xi_i \right) = \sum_i \eta_i = \sum_i \beta_i \eta = \eta.$$

Thus for $\xi = \sum_i \xi_i$, we have $D_u \xi = \eta$. In order to prove that the mapping i is injective immersion we need to prove the analogous estimate as in (4.51), more precisely we prove that each $\xi \in W_{bc}^{2,2}(\Sigma_0, u^*TM)$ satisfies the inequality

$$\|\xi\|_{2,2} \leq c \left(\|D_u \xi\|_{1,2} + \|\xi \circ \iota_{i_0}^{\beta,+}(s_0, \cdot)\|_{3/2} \right). \quad (5.46)$$

Let β_i be the partition of unity as above and let $\xi_i = \beta_i \xi$. As in the proof of the inequality (4.51) (have a look at Lemma 3.3.6) we have that each ξ_i satisfies the following inequality

$$\|\xi_i\|_{2,2} \leq c \left(\|D_u \xi_i\|_{1,2} + \|K_i \xi_i\|_{1,2} + \|\xi_i \circ \iota_{i_0}^{\beta,+}(s_0, \cdot)\|_{3/2} \right),$$

where K_i are some compact operators (for example just a restriction operator to some compact set). Summing these inequalities for all i we obtain

$$\begin{aligned} \|\xi\|_{2,2} &\leq c \left(\sum_i \|D_u(\beta_i \xi)\|_{1,2} + \sum_i \|K_i \xi_i\|_{1,2} + \|\xi \circ \iota_{i_0}^{\beta,+}(s_0, \cdot)\|_{3/2} \right) \\ &\leq c \left(\sum_i \|\beta_i D_u(\xi)\|_{1,2} + \sum_i (\|\dot{\beta}_i \xi\|_{1,2} + \|K_i \xi_i\|_{1,2}) + \|\xi \circ \iota_{i_0}^{\beta,+}(s_0, \cdot)\|_{3/2} \right) \\ &\leq c \left(\|D_u \xi\|_{1,2} + \|K \xi\|_{1,2} + \|\xi \circ \iota_{i_0}^{\beta,+}(s_0, \cdot)\|_{3/2} \right), \end{aligned}$$

where K is some compact operator. From unique continuation we have that the mapping $\xi \mapsto (D_u \xi, \xi \circ \iota_{i_0}^{\beta,+}(s_0, \cdot))$ is injective and as the above inequality holds its image is closed, thus from the open mapping theorem its inverse is also bounded and we can omit the middle term from the above inequality. Thus, we have proved the inequality (5.46). This inequality implies that the mapping i is an immersion, and from unique continuation of perturbed holomorphic curves follows that i is also injective. \square

5.6 Proof of the Floer Gluing Theorems

Boundary Operator

In this section we prove Theorem 5.4.1.

Lemma 5.6.1. *Assume (H) and let (H, J) be a regular pair for (L_0, L_1) . Let $x, y \in \mathcal{C}(L_0, L_1; H)$ and fix a real number $\delta > 0$. Then, for every Floer trajectory $u \in \mathcal{M}(x, y; H, J)$ with $E_H(u) > \delta$, there is a unique real number $T_\delta(u)$ such that*

$$\int_{-\infty}^{T_\delta(u)} \int_0^1 |\partial_s u|_t^2 dt ds = \delta.$$

Moreover, the map

$$T_\delta : \{u \in \mathcal{M}(x, y; H, J) \mid E_H(u) > \delta\} \rightarrow \mathbb{R}$$

is smooth.

Proof. Define the map $E_H : \mathcal{M}(x, y; H, J) \times \mathbb{R} \rightarrow (0, \infty)$ by

$$E_H(u, T) := \int_{-\infty}^T \int_0^1 |\partial_s u|_t^2 dt ds.$$

This map is smooth and

$$\frac{\partial}{\partial T} E_H(u, T) = \int_0^1 |\partial_s u(T, t)|_t^2 dt$$

for every $u \in \mathcal{M}(x, y; H, J)$ and every $T \in \mathbb{R}$. By unique continuation we have $\partial_T E_H(u, T) > 0$ for every $u \in \mathcal{M}(x, y; H, J)$ and every $T \in \mathbb{R}$ such that $E_H(u, T) > 0$. Hence the assertions of Lemma 5.6.1 follow from the intermediate value theorem (existence), the fact that the map $T \mapsto E_H(u, T)$ is strictly monotone unless $\partial_s u \equiv 0$ (uniqueness), and the implicit function theorem (smoothness). \square

5.6.2. Let $\mathcal{P} = \mathcal{P}^{3/2}$ be the path space defined in equation (4.9). Choose open neighborhoods $U, V \subset M$ of $y(0)$ as in 4.1.6, choose a constant $\hbar > 0$ such that the assertion of Theorem 2.1.4 holds with $U, V, \Lambda = \{y(0)\}$, and choose a neighborhood $\mathcal{U} \subset \mathcal{P}$ of y and a constant $T_0 > 0$ such that the assertions of Theorem 4.1.8 are satisfied with x replaced by y . Here the Hilbert manifolds $\mathcal{M}^\infty(y, \mathcal{U})$ and $\mathcal{M}^T(y, \mathcal{U})$ are defined by (4.16) and the embeddings

$$\iota^\infty : \mathcal{M}^\infty(y, \mathcal{U}) \rightarrow \mathcal{P} \times \mathcal{P}, \quad \iota^T : \mathcal{M}^T(y, \mathcal{U}) \rightarrow \mathcal{P} \times \mathcal{P}$$

for $T \geq T_0$ are defined by (4.15) and (4.13).

Proof of Theorem 5.4.1. The proof has five steps.

Step 1. *Abbreviate*

$$u^\pm := u|_{\mathbb{R}^\pm \times [0, 1]}, \quad v^\pm := v|_{\mathbb{R}^\pm \times [0, 1]}.$$

We may assume without loss of generality that

$$\delta_u := E_H(u^+) < \hbar/2, \quad \delta_v := E_H(v^-) < \hbar/2, \quad u(0, \cdot), v(0, \cdot) \in \mathcal{U}. \quad (5.47)$$

Choose $s_0 > 0$ so large that the solutions $\tilde{u}, \tilde{v} : \mathbb{R} \times [0, 1] \rightarrow M$ of the Floer equations, defined by

$$\begin{aligned}\tilde{u}(s, t) &:= u(s + s_0, t), & \tilde{s}_u &:= s_u - s_0, \\ \tilde{v}(s, t) &:= v(s - s_0, t), & \tilde{s}_v &:= s_v + s_0,\end{aligned}$$

satisfy the conditions in (5.47). Assume that Theorem 5.4.1 holds for the quadruple $(\tilde{u}, \tilde{v}, \tilde{s}_u, \tilde{s}_v)$ and denote the resulting glued solutions of the Floer equation by $\tilde{u}_T \in \mathcal{M}^2(x, z; H, J)$ for $T > \tilde{T}_0$. Then the functions

$$u_T(s, t) := \tilde{u}_{T-s_0}(s, t), \quad T > T_0 := \tilde{T}_0 + s_0$$

satisfy the requirements of Theorem 5.4.1 for the quadruple (u, v, s_u, s_v) .

Step 2. *Construction of the map $(T_0, \infty) \rightarrow \mathcal{M}^2(x, z; H, J) : T \mapsto u_T$.*

By Step 1, we have $(u^+, v^-) \in \mathcal{M}^\infty(y, \mathcal{U})$ and

$$(u(0, \cdot), v(0, \cdot)) = \iota^\infty(u^+, v^-) \in \mathcal{W}^\infty(y, \mathcal{U}).$$

Now define

$$\varepsilon^- := E_H(u^-) = E_H(u) - \delta_u, \quad \varepsilon^+ := E_H(v^+) = E_H(v) - \delta_v.$$

Then the spaces

$$\begin{aligned}\mathcal{M}^-(x, \varepsilon^-) &:= \left\{ w^- \in W_{\text{loc}}^{2,2}(\mathbb{R}^- \times [0, 1], M) \left| \begin{array}{l} w^- \text{ satisfies (5.1),} \\ E_H(w^-) = \varepsilon^-, \\ \lim_{s \rightarrow -\infty} w^-(s, t) = x(t) \end{array} \right. \right\}, \\ \mathcal{M}^+(z, \varepsilon^+) &:= \left\{ w^+ \in W_{\text{loc}}^{2,2}(\mathbb{R}^- \times [0, 1], M) \left| \begin{array}{l} w^+ \text{ satisfies (5.1),} \\ E_H(w^+) = \varepsilon^+, \\ \lim_{s \rightarrow \infty} w^+(s, t) = z(t) \end{array} \right. \right\}\end{aligned}$$

are Hilbert manifolds and the restriction maps

$$\iota^- : \mathcal{M}^-(x, \varepsilon^-) \rightarrow \mathcal{P}, \quad \iota^+ : \mathcal{M}^+(z, \varepsilon^+) \rightarrow \mathcal{P}$$

defined by $\iota^-(w^-) := w^-(0, \cdot)$ and $\iota^+(w^+) := w^+(0, \cdot)$ are injective immersions. The transversality condition asserts that the map

$$\iota^- \times \iota^+ : \mathcal{M}^-(x, \varepsilon^-) \times \mathcal{M}^+(z, \varepsilon^+) \rightarrow \mathcal{P} \times \mathcal{P} \quad (5.48)$$

intersects the submanifold

$$\mathcal{W}^\infty(y, \mathcal{U}) = \iota^\infty(\mathcal{M}^\infty(y, \mathcal{U})) \subset \mathcal{P} \times \mathcal{P}$$

transversally in the point $(u(0, \cdot), v(0, \cdot)) = (\iota^-(u^-), \iota^+(v^+))$. Moreover, the Fredholm index is zero, so the intersection point is isolated. Hence, by Theorem 4.1.8, the map (5.48) is also transverse to the submanifold

$$\mathcal{W}^T(y, \mathcal{U}) = \iota^T(\mathcal{M}^T(y, \mathcal{U})) \subset \mathcal{P} \times \mathcal{P}$$

for T sufficiently large. Hence it follows from the infinite dimensional inverse function theorem that there is a $T_0 > 0$ such that, for every $T > T_0$, there exists a unique pair $(u_T^-, v_T^+) \in \mathcal{M}^-(x, \varepsilon^-) \times \mathcal{M}^+(z, \varepsilon^+)$ near the pair (u^-, v^+) such that $(\iota^-(u_T^-), \iota^+(v_T^+)) \in \mathcal{W}^T(y, \mathcal{U})$. Thus, for $T > T_0$, there is a unique element $w_T \in \mathcal{M}^T(y, \mathcal{U})$ such that

$$w_T(-T, t) = u_T^-(0, t), \quad w_T(T, t) = v_T^+(0, t)$$

for every $t \in [0, 1]$. Now define $u_T : \mathbb{R} \times [0, 1] \rightarrow M$ by

$$u_T(s, t) := \begin{cases} u_T^-(s + T, t), & \text{if } s \leq -T, \\ w_T(s, t), & \text{if } |s| \leq T, \\ v_T^+(s - T, t), & \text{if } s \geq T. \end{cases} \quad (5.49)$$

Then $u_T \in \mathcal{M}^2(x, z; H, J)$ for every $T \geq T_0$. This proves Step 2.

Step 3. *The map $T \mapsto u_T$ satisfies (i) and (iii).*

For $w \in \mathcal{M}^2(x, z; H, J)$ denote its equivalence class under time shift by $[w]$. Define the map $T : \mathcal{M}^2(x, z; H, J) \rightarrow (0, \infty)$ by

$$T(w) := \frac{T_{\varepsilon^- + \delta_u + \delta_v}(w) - T_{\varepsilon^-}(w)}{2}$$

for $w \in \mathcal{M}^2(x, z; H, J)$. This map is smooth, by Lemma 5.6.1, and is invariant under time shift. Hence the map descends to the quotient $\widehat{\mathcal{M}}^2(x, z; H, J)$. Moreover, by construction

$$E_H(u_T) = E_H(u) + E_H(v) = \varepsilon^- + \delta_u + \delta_v + \varepsilon^+. \quad (5.50)$$

The energy of u_T on $(-\infty, T] \times [0, 1]$ is equal to ε^- and the energy of u_T on $[T, \infty) \times [0, 1]$ is equal to ε^+ . Hence $T_{\varepsilon^- + \delta_u + \delta_v}(u_T) = T$ and $T_{\varepsilon^-}(u_T) = -T$, and hence $T(u_T) = T$ for every $T > T_0$. This shows that the map

$$(T_0, \infty) \rightarrow \widehat{\mathcal{M}}^2(x, z; H, J) : T \mapsto [u_T]$$

is injective, its image is an open subset of the 1-manifold $\widehat{\mathcal{M}}^2(x, z; H, J)$, and its inverse is smooth by Lemma 5.6.1. Thus we have proved Step 3.

Step 4. *The map $T \mapsto u_T$ satisfies (ii).*

By construction in Step 2, the functions u_T^- converge to $u^- := u|_{\mathbb{R}^- \times [0,1]}$ in the Hilbert manifold $\mathcal{M}^-(x, \varepsilon^-)$ as T tends to infinity, and the functions v_T^+ converge to $v^+ := v|_{\mathbb{R}^+ \times [0,1]}$ in the Hilbert manifold $\mathcal{M}^+(z, \varepsilon^+)$ as T tends to infinity. (See the proof of Step 2.) (It follows also from the construction that $u_T([-T, T] \times \{t\})$ is contained in the neighborhood $U_t = \phi_t(U)$ of $y(t)$ for every $t \in [0, 1]$ and every $T > T_0$.) Hence $u_T(s-T, t) = u_T^-(s, t)$ converges to $u(s, t)$ in the $W^{2,2}$ topology on $\mathbb{R}^- \times [0, 1]$ and $u_T(s+T, t) = u_T^+(s, t)$ converges to $v(s, t)$ in the $W^{2,2}$ topology on $\mathbb{R}^+ \times [0, 1]$. By elliptic bootstrapping it follows that the functions $u_T(-T + \cdot, \cdot)$ converges to u uniformly with all derivatives on every compact subset of $(-\infty, 0) \times [0, 1]$. Likewise, the functions $u_T(T + \cdot, \cdot)$ converge to v uniformly with all derivatives on every compact subset of $(0, \infty) \times [0, 1]$. Since no energy is lost, by (ii), one can now use the standard compactness theorem for Floer trajectories to exclude bubbling and prove in both cases that the convergence is in the C^∞ topology on every compact subset of $\mathbb{R} \times [0, 1]$. The above convergence statement in the Hilbert manifold $\mathcal{M}^-(x, \varepsilon^-)$ also implies that $u_T(s-T, t)$ converges to $u(s, t)$ uniformly on $\mathbb{R}^- \times [0, 1]$ and hence on every subset of the form $(-\infty, b] \times [0, 1]$. Likewise, $u_T(s+T, t)$ converges to $v(s, t)$ uniformly on every subset of the form $[a, \infty) \times [0, 1]$.

Step 5. *The map $T \mapsto u_T$ satisfies (iv).*

Assume, by contradiction, that (iv) does not hold. Then there are sequences $w_i \in \mathcal{M}^2(x, z; H, J)$ and $s_i^- < s_i^+$ such that

- (a) $[w_i] \neq [u_T]$ for every i and every $T > T_0$,
- (b) $E_H(w_i) = E_H(u) + E_H(v)$ for every i , and
- (c) For $0 \leq t \leq 1$ we have

$$\lim_{i \rightarrow \infty} w_i(s_i^-, t) = u(s_u, t), \quad \lim_{i \rightarrow \infty} w_i(s_i^+, t) = v(s_v, t).$$

The convergence is uniform in t .

By the standard elliptic bootstrapping, bubbling, and removal of singularities argument, we may assume, passing to a subsequence if necessary, that the sequence $w_i(s_i^\pm + \cdot, \cdot)$ converges, uniformly with all derivatives on every compact subset of the complement of a finite set in $\mathbb{R} \times [0, 1]$, to a smooth finite energy solution $u^\pm : \mathbb{R} \times [0, 1] \rightarrow M$ of (5.1). (See [16, Chapter 4].) By (c) we have

$$u^-(0, t) = u(s_u, t), \quad u^+(0, t) = v(s_v, t)$$

for every $t \in [0, 1]$. Hence it follows from unique continuation that

$$u^-(s, t) = u(s_u + s, t), \quad u^+(s, t) = v(s_v + s, t)$$

for all s and t . Since u and v are not related by time shift, it follows that $s_i^+ - s_i^-$ diverges to ∞ .

Moreover, it follows from (b) that there is no loss of energy. Hence there is no bubbling and

$$\begin{aligned} u(s, t) &= \lim_{i \rightarrow \infty} w_i(r_i^- + s, t), & r_i^- &:= s_i^- - s_u, \\ v(s, t) &= \lim_{i \rightarrow \infty} w_i(r_i^+ + s, t), & r_i^+ &:= s_i^+ + s_v. \end{aligned} \quad (5.51)$$

The convergence is uniform with all derivatives on every compact subset of $\mathbb{R} \times [0, 1]$. This implies that, for every $T > 0$,

$$\begin{aligned} \varepsilon_T &:= E_H(u) + E_H(v) - \int_{-T}^T \int_0^1 |\partial_s u|_t^2 - \int_{-T}^T \int_0^1 |\partial_s v|_t^2 \\ &= \lim_{i \rightarrow \infty} \left(E_H(w_i) - \int_{r_i^- - T}^{r_i^- + T} \int_0^1 |\partial_s w_i|_t^2 - \int_{r_i^+ - T}^{r_i^+ + T} \int_0^1 |\partial_s w_i|_t^2 \right) \\ &= \lim_{i \rightarrow \infty} \left(\int_{-\infty}^{r_i^- - T} \int_0^1 |\partial_s w_i|_t^2 + \int_{r_i^- + T}^{r_i^+ - T} \int_0^1 |\partial_s w_i|_t^2 + \int_{r_i^+ + T}^{\infty} \int_0^1 |\partial_s w_i|_t^2 \right). \end{aligned}$$

Here we have used (b). Taking the limit $T \rightarrow \infty$, we deduce that all three limits on the right converge to zero as T tends to infinity. Hence

$$\begin{aligned} \varepsilon^- &= \int_{-\infty}^0 \int_0^1 |\partial_s u|_t^2 = \lim_{T \rightarrow \infty} \int_{-T}^0 \int_0^1 |\partial_s u|_t^2 = \lim_{T \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{r_i^- - T}^{r_i^-} \int_0^1 |\partial_s w_i|_t^2 \\ &= \lim_{T \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{r_i^- - T}^{r_i^-} \int_0^1 |\partial_s w_i|_t^2 + \lim_{T \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{-\infty}^{r_i^- - T} \int_0^1 |\partial_s w_i|_t^2 \\ &= \lim_{i \rightarrow \infty} \int_{-\infty}^{r_i^-} \int_0^1 |\partial_s w_i|_t^2, \end{aligned}$$

and similarly $\varepsilon^+ = \lim_{i \rightarrow \infty} \int_{r_i^+}^{\infty} \int_0^1 |\partial_s w_i|_t^2$. Adding to r_i^\pm a sequence converging to zero, if necessary, we may assume w.l.o.g. that, for every i ,

$$\varepsilon^- = \int_{-\infty}^{r_i^-} \int_0^1 |\partial_s w_i|_t^2, \quad \varepsilon^+ = \int_{r_i^+}^{\infty} \int_0^1 |\partial_s w_i|_t^2. \quad (5.52)$$

We claim that, for large i ,

$$w_i(T_i + \cdot, \cdot) = u_{T_i}, \quad T_i := \frac{1}{2}(r_i^+ - r_i^-) \quad (5.53)$$

for i sufficiently large, in contradiction to (a). To see this, note that by (5.52),

$$\begin{aligned} u_i^- &:= w_i(r_i^- + \cdot, \cdot)|_{\mathbb{R}^- \times [0,1]} \in \mathcal{M}^-(x, \varepsilon^-), \\ v_i^+ &:= w_i(r_i^+ + \cdot, \cdot)|_{\mathbb{R}^+ \times [0,1]} \in \mathcal{M}^+(z, \varepsilon^+). \end{aligned}$$

Then, by (5.51) and (5.52), the sequence u_i^- converges to u^- in $\mathcal{M}^-(x, \varepsilon^-)$ and the sequence v_i^+ converges to v^+ in $\mathcal{M}^+(z, \varepsilon^+)$. Moreover, by (b) and (5.52), we have $\int_{r_i^-}^{r_i^+} \int_0^1 |\partial_s w_i|^2_t = \delta_u + \delta_v < \hbar$ for every i . Since

$$\lim_{i \rightarrow \infty} w_i(r_i^-, \cdot) = u(0, \cdot) \in \mathcal{U}, \quad \lim_{i \rightarrow \infty} w_i(r_i^+, \cdot) = v(0, \cdot) \in \mathcal{U},$$

where the convergence is in the topology of $\mathcal{P} = \mathcal{P}^{3/2}$, it follows from the definition of \hbar in the proof of Step 2, that

$$(\iota^-(u_i^-), \iota^+(v_i^+)) = (w_i(r_i^-, \cdot), w_i(r_i^+, \cdot)) \in \mathcal{W}^{T_i}(y, \mathcal{U})$$

for i sufficiently large. Hence it follows from the definition of u_T in (5.49) that

$$u_{T_i}(s, t) := \begin{cases} u_i^-(s + T_i, t), & \text{if } s \leq -T_i, \\ w_i(\frac{r_i^+ + r_i^-}{2} + s, t), & \text{if } |s| \leq T_i, \\ v_i^+(s - T_i, t), & \text{if } s \geq T_i, \end{cases} = w_i(T_i + s, t).$$

This proves equation (5.53) and Theorem 5.4.1. \square

Chain Map

In this section we prove Theorem 5.4.3.

5.6.3. Let (Σ, L) be a string cobordism in $\mathcal{L}(M, \omega)$ from (L_0^α, L_1^α) to (L_0^β, L_1^β) , (j, H, J) be regular set of Floer data on (Σ, L) from (H^α, J^α) to (H^β, J^β) , and

$$u \in \mathcal{M}^0(x^\alpha, y^\beta; j, H, J), \quad v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta),$$

as in the assumptions of Theorem 5.4.3. Thus

$$v = \{v_i\}_{i \in I^\beta}, \quad v_i \in \mathcal{M}(y_i^\beta, z_i^\beta; H_i^\beta, J_i^\beta),$$

and there is an index $i_0 \in I^\beta$ such that $v_i(s, t) = y_i^\beta(t) = z_i^\beta(t)$ for $i \neq i_0$ and $\mu_H(v_{i_0}) = 1$.

5.6.4. As in 5.6.2, let $\mathcal{P} = \mathcal{P}^{3/2}$ be the path space defined in equation (4.34). Choose open neighborhoods $U, V \subset M$ of $y_{i_0}^\beta(0)$ as in 4.1.6, choose a constant $\hbar > 0$ such that the assertion of Theorem 2.1.4 holds with $U, V, \Lambda = \{y_{i_0}^\beta(0)\}$, and choose a neighborhood $\mathcal{U} \subset \mathcal{P}$ of $y_{i_0}^\beta$ and a constant $T_0 > 0$ such that the assertions of Theorem 4.1.8 are satisfied with x replaced by $y_{i_0}^\beta$. Here the Hilbert manifolds $\mathcal{M}^\infty(y_{i_0}^\beta, \mathcal{U})$ and $\mathcal{M}^T(y_{i_0}^\beta, \mathcal{U})$ are defined by (4.16) and the embeddings

$$\iota^\infty : \mathcal{M}^\infty(y_{i_0}^\beta, \mathcal{U}) \rightarrow \mathcal{P} \times \mathcal{P}, \quad \iota^T : \mathcal{M}^T(y_{i_0}^\beta, \mathcal{U}) \rightarrow \mathcal{P} \times \mathcal{P}$$

for $T \geq T_0$ are defined by (4.15) and (4.13). Denote their images by

$$\mathcal{W}^\infty(y_{i_0}^\beta, \mathcal{U}) := \iota^\infty(\mathcal{M}^\infty(y_{i_0}^\beta, \mathcal{U})), \quad \mathcal{W}^T(y_{i_0}^\beta, \mathcal{U}) := \iota^T(\mathcal{M}^T(y_{i_0}^\beta, \mathcal{U})).$$

Proof of Theorem 5.4.3. The proof has three steps.

Step 1. *Construction of the map $(T_0, \infty) \rightarrow \mathcal{M}^1(x^\alpha, z^\beta; j, H, J) : T \mapsto u_T$.*

Choose $s_0 > 0$ so large that

$$\begin{aligned} \int_{s_0}^\infty \int_0^1 \left| \partial_s u(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds &< \hbar/2, \\ \delta_v &:= \int_{-\infty}^{-s_0} \int_0^1 \left| \partial_s v(s, t) \right|_t^2 dt ds < \hbar/2, \\ u(\iota_{i_0}^{\beta,+}(s_0, \cdot)) &\in \mathcal{U}, \quad v_{i_0}(-s_0, \cdot) \in \mathcal{U}. \end{aligned} \tag{5.54}$$

Then

$$(u(\iota_{i_0}^{\beta,+}(s_0, \cdot)), v_{i_0}(-s_0, \cdot)) \in \mathcal{W}^\infty(y_{i_0}^\beta, \mathcal{U}).$$

Denote

$$\Sigma_0 := \Sigma \setminus \iota_{i_0}^{\beta,+}((s_0, \infty) \times [0, 1]), \quad \varepsilon_v := E_H(v) - \delta_v.$$

Then the spaces

$$\begin{aligned} \mathcal{M}^- &:= \left\{ w^- \in W_{\text{loc}}^{2,2}(\Sigma_0, M) \left| \begin{array}{l} w^- \text{ satisfies (5.23), } E_H(w^-) < \infty, \\ \lim_{s \rightarrow -\infty} w^-(\iota_i^{\alpha,-}(s, t)) = x_i^\alpha(t), i \in I^\alpha \\ \lim_{s \rightarrow \infty} w^-(\iota_i^{\beta,+}(s, t)) = y_i^\beta(t), i \in I^\beta \setminus \{i_0\} \end{array} \right. \right\}, \\ \mathcal{M}^+ &:= \left\{ w^+ \in W_{\text{loc}}^{2,2}([-s_0, \infty) \times [0, 1], M) \left| \begin{array}{l} w^+ \text{ satisfies (5.1),} \\ E_H(w^+) = \varepsilon_v, \\ \lim_{s \rightarrow \infty} w^+(s, t) = z_{i_0}^\beta(t) \end{array} \right. \right\} \end{aligned}$$

are Hilbert manifolds and the restriction maps $\iota^\pm : \mathcal{M}^\pm \rightarrow \mathcal{P}$, defined by

$$\iota^-(w^-) := w^-(\iota_{i_0}^{\beta,+}(s_0, \cdot)), \quad \iota^+(w^+) := w^+(-s_0, \cdot),$$

are injective immersions (see 4.1.5). By transversality, the map

$$\iota^- \times \iota^+ : \mathcal{M}^- \times \mathcal{M}^+ \rightarrow \mathcal{P} \times \mathcal{P} \quad (5.55)$$

intersects the submanifold $\mathcal{W}^\infty(y_{i_0}^\beta, \mathcal{U})$ transversally in the point

$$(u(\iota_{i_0}^{\beta,+}(s_0, \cdot)), v(-s_0, \cdot)) = (\iota^-(u|_{\Sigma_0}), \iota^+(v|_{[-s_0, \infty) \times [0, 1]})).$$

Moreover, the Fredholm index is zero, so the intersection point is isolated. Hence, by Theorem 4.1.8, the map (5.55) is also transverse to the submanifold $\mathcal{W}^T(y_{i_0}^\beta, \mathcal{U})$ for T sufficiently large. Hence it follows from the infinite dimensional inverse function theorem that there is a $T_0 > 0$ such that, for every $T > T_0$, there exists a unique pair $(u_T^-, v_T^+) \in \mathcal{M}^- \times \mathcal{M}^+$ near the pair $(u|_{\Sigma_0}, v|_{[-s_0, \infty) \times [0, 1]})$ such that

$$(\iota^-(u_T^-), \iota^+(v_T^+)) \in \mathcal{W}^T(y, \mathcal{U}).$$

Thus, for $T > T_0$, there is a unique element $w_T \in \mathcal{M}^T(y_{i_0}^\beta, \mathcal{U})$ such that

$$w_T(-T, t) = u_T^-(\iota_{i_0}^{\beta,+}(s_0, t)), \quad w_T(T, t) = v_T^+(-s_0, t)$$

for every $t \in [0, 1]$. Now define $u_T : \mathbb{R} \times [0, 1] \rightarrow M$ by

$$\begin{cases} u_T(z) := u_T^-(z), & \text{if } z \in \Sigma_0, \\ u_T(\iota_{i_0}^{\beta,+}(s_0 + s, t)) := w_T(-T + s, t), & \text{if } |s| \leq T, \\ u_T(\iota_{i_0}^{\beta,+}(s_0 + 2T + s, t)) := v_T^+(-s_0 + s, t), & \text{if } s \geq 0. \end{cases} \quad (5.56)$$

Then $u_T \in \mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$ for every $T \geq T_0$. This proves Step 1.

Step 2. *The map $T \mapsto u_T$ satisfies (i), (ii), and (iii).*

It follows directly from the construction and the homotopy invariance of the integral of the pullback of ω under maps with Lagrangian boundary conditions that

$$\int_{\Sigma} u_T^* \omega = \int_{\Sigma} u^* \omega + \int_{\mathbb{R} \times [0, 1]} v^* \omega$$

for every $T > T_0$. Hence $\mathcal{A}_H(u_T) = \mathcal{A}_H(u) + \mathcal{A}_H(v)$ for every $T \geq T_0$ and this proves (iii). Consider the open subset

$$\mathcal{M}_{\varepsilon_v} := \left\{ w \in \mathcal{M}^1(x^\alpha, z^\beta; j, H, J) \mid \int_{s_0}^{\infty} \int_0^1 \left| w(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds > \varepsilon_v \right\}.$$

Then, for each $w \in \mathcal{M}_{\varepsilon_v}$, there is a unique $T = T(w) > 0$ such that

$$\int_{s_0+2T}^{\infty} \int_0^1 \left| w(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds = \varepsilon_v$$

By Lemma 5.6.1 the map $\mathcal{M}_{\varepsilon_v} \rightarrow (0, \infty) : w \mapsto T(w)$ is smooth. Moreover, by construction we have $u_T \in \mathcal{M}_{\varepsilon_v}$ and $T(u_T) = T$ for every $T \geq T_0$. Hence the map $T \mapsto u_T$ is a diffeomorphism onto its image and this proves (i). The proof that the map $T \mapsto u_T$ satisfies (ii) is almost verbatim the same as the proof of Step 4 in the proof of Theorem 5.4.1 and will be omitted. Thus we have proved Step 2.

Step 3. *The map $T \mapsto u_T$ satisfies (iv).*

Assume, by contradiction, that (iv) does not hold. Then there are sequences $w_i \in \mathcal{M}^1(x^\alpha, z^\beta; j, H, J)$ and $s_i \geq 0$ such that

- (a) $w_i \neq u_T$ for every i and every $T > T_0$,
- (b) $\mathcal{A}_H(w_i) = \mathcal{A}_H(u) + \mathcal{A}_H(v)$ for every i , and
- (c) $\limsup_{i \rightarrow \infty} d(w_i, u) = 0$ and $\limsup_{i \rightarrow \infty} \sup_{0 \leq t \leq 1} d(w_i(\iota_{i_0}^{\beta,+}(s_i, t)), v(s_v, t)) = 0$.

By the standard elliptic bootstrapping, bubbling, and removal of singularities argument, we may assume, passing to a subsequence if necessary, that the sequence w_i converges uniformly with all derivatives on every compact subset of the complement of a finite set in Σ to a smooth finite energy solution $\tilde{u} : \Sigma \rightarrow M$ of (5.23), and the sequence $w_i(\iota_{i_0}^{\beta,+}(s_i + \cdot, \cdot))$ converges, uniformly with all derivatives on every compact subset of the complement of a finite set in $\mathbb{R} \times [0, 1]$, to a smooth finite energy solution $\tilde{v} : \mathbb{R} \times [0, 1] \rightarrow M$ of (5.1) (See [16, Chapter 4].) By (c) we have $\tilde{u}(z) = u(z)$ for every $z \in W_0$ and $\tilde{v}(0, t) = v(s_v, t)$ for every $t \in [0, 1]$. Hence it follows from unique continuation that $\tilde{u} = u$ and $\tilde{v}(s, t) = v(s_v + s, t)$ for all s and t .

Next we claim that s_i diverges to infinity. Otherwise, by passing to a further subsequence, it would follow that $s_i \rightarrow s^*$ and hence

$$u(\iota_{i_0}^{\beta,+}(s^*, t)) = \lim_{i \rightarrow \infty} w_i(\iota_{i_0}^{\beta,+}(s_i, t)) = v(s_v, t).$$

By unique continuation it would follow that $u(\iota_{i_0}^{\beta,+}(s^* + s, t)) = v(s_v + s, t)$ for every $s \geq 0$ contradicting the fact that

$$\lim_{s \rightarrow \infty} u(\iota_{i_0}^{\beta,+}(s^* + s, \cdot)) = y_{i_0}^\beta \neq z_{i_0}^\beta = \lim_{s \rightarrow \infty} v(s_v + s, \cdot).$$

This shows that $s_i \rightarrow \infty$, as claimed.

Next we prove that

$$\lim_{i \rightarrow \infty} \int_{r_i}^{\infty} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds = \varepsilon_v, \quad r_i := s_i - s_v + s_0. \quad (5.57)$$

First, it follows from (b) that there is no loss of energy. Hence there is no bubbling and

$$u(z) = \lim_{i \rightarrow \infty} w_i(z), \quad v(s_0 + s, t) = \lim_{i \rightarrow \infty} w_i(\iota_{i_0}^{\beta,+}(r_i + s, t)) \quad (5.58)$$

The convergence is uniform with all derivatives on every compact subset of Σ , respectively $\mathbb{R} \times [0, 1]$. This implies that, for every $T > 0$,

$$\lim_{i \rightarrow \infty} \int_{r_i}^{r_i+T} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds = \int_0^T \int_0^1 |v(s_0 + s, t)|_t^2 dt ds.$$

Taking the limit $T \rightarrow \infty$ we obtain

$$\begin{aligned} \varepsilon_v &= \lim_{T \rightarrow \infty} \int_0^T \int_0^1 |v(s_0 + s, t)|_t^2 dt ds \\ &= \lim_{T \rightarrow \infty} \lim_{i \rightarrow \infty} \int_{r_i}^{r_i+T} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds \\ &\leq \lim_{i \rightarrow \infty} \int_{r_i}^{\infty} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds. \end{aligned}$$

If the inequality is strict it would follow that $\mathcal{A}_H(w_i) > \mathcal{A}_H(u) + \mathcal{A}_H(v)$ for large i , contradicting (b). Thus we have proved (5.57). It follows from (5.58) that, for i sufficiently large, we have

$$z_{i_0}^{\beta}(t) = \lim_{s \rightarrow \infty} w_i(\iota_{i_0}^{\beta,+}(s, t)).$$

Moreover, passing to a further subsequence and adding to r_i a sequence converging to zero, if necessary, we may assume w.l.o.g. that

$$\int_{r_i}^{\infty} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds = \varepsilon_v \quad (5.59)$$

for every i .

Next we claim that, for large i ,

$$w_i = u_{T_i}, \quad T_i := \frac{1}{2} (r_i - s_0) \quad (5.60)$$

for i sufficiently large, in contradiction to (a). To see this, note that by (5.59),

$$\begin{aligned} u_i^- &:= w_i|_{\Sigma_0} \in \mathcal{M}^-, \\ v_i^+ &:= w_i \circ \iota_{i_0}^{\beta,+}(s_0 + r_i + \cdot, \cdot)|_{[-s_0, \infty) \times [0, 1]} \in \mathcal{M}^+. \end{aligned}$$

Then, by (5.58) and (5.59), the sequence u_i^- converges to u^- in \mathcal{M}^- and the sequence v_i^+ converges to v^+ in \mathcal{M}^+ . Moreover, by (b) and (5.59), we have

$$\begin{aligned} \int_{s_0}^{r_i} \int_0^1 |\partial_s w_i|_t^2 &= \mathcal{A}_H(w_i) - \mathcal{A}_H(w_i|_{\Sigma_0}) - \int_{r_i}^{\infty} \int_0^1 \left| w_i(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds \\ &= \mathcal{A}_H(w_i) - \mathcal{A}_H(w_i|_{\Sigma_0}) - \varepsilon_v \\ &= \mathcal{A}_H(u) + \mathcal{A}_H(v) - \mathcal{A}_H(w_i|_{\Sigma_0}) - \varepsilon_v \\ &\rightarrow \mathcal{A}_H(u) - \mathcal{A}_H(u|_{\Sigma_0}) + \mathcal{A}_H(v) - \varepsilon_v \\ &= \int_{s_0}^{\infty} \int_0^1 \left| u(\iota_{i_0}^{\beta,+}(s, t)) \right|_t^2 dt ds + \delta_v \\ &< \hbar. \end{aligned}$$

Here the arrow denotes the limit $i \rightarrow \infty$ and the last inequality follows from (5.54). By (5.58), we have

$$\begin{aligned} \lim_{i \rightarrow \infty} w_i(\iota_{i_0}^{\beta,+}(s_0, \cdot)) &= u(\iota_{i_0}^{\beta,+}(s_0, \cdot)) \in \mathcal{U}, \\ \lim_{i \rightarrow \infty} w_i(\iota_{i_0}^{\beta,+}(r_i, \cdot)) &= v(-s_0, \cdot) \in \mathcal{U}. \end{aligned}$$

Here the convergence is in the C^∞ topology and hence also in the topology of $\mathcal{P} = \mathcal{P}^{3/2}$. Hence it follows from the definition of \hbar that

$$(\iota^-(u_i^-), \iota^+(v_i^+)) = \left(w_i(\iota_{i_0}^{\beta,+}(s_0, \cdot)), w_i(\iota_{i_0}^{\beta,+}(r_i, \cdot)) \right) \in \mathcal{W}^{T_i}(y_{i_0}^\beta, \mathcal{U})$$

for i sufficiently large. Hence it follows from the definition of u_T in (5.56) that $w_i = u_{T_i}$ for i sufficiently large. This proves (5.60) and Theorem 5.4.3. \square

Chain Homotopy Equivalence

In this section we prove Theorem 5.4.5.

5.6.5. Let (Σ, L) be a string cobordism in $\mathcal{L}(M, \omega)$ from (L_0^α, L_1^α) to (L_0^β, L_1^β) , (j_0, H_0, J_0) and (j_1, H_1, J_1) be two regular sets of Floer data on (Σ, L) from (H^α, J^α) to (H^β, J^β) , and let $\{j_\lambda, H_\lambda, J_\lambda\}_{0 \leq \lambda \leq 1}$ be a regular homotopy of Floer data from (j_0, H_0, J_0) to (j_1, H_1, J_1) . Let

$$0 < \lambda_\infty < 1, \quad u \in \mathcal{M}^{-1}(x^\alpha, y^\beta; j_{\lambda_\infty}, H_{\lambda_\infty}, J_{\lambda_\infty})$$

and

$$v \in \mathcal{M}^1(y^\beta, z^\beta; H^\beta, J^\beta)$$

be as in the assumptions of Theorem 5.4.5. Thus there exists an index $i_0 \in I^\beta$ such that

$$v_i(s, t) = y_i^\beta(t) = z_i^\beta(t)$$

for $i \neq i_0$ and

$$\mu_H(v_{i_0}) = 1.$$

Choose neighborhoods U, V of $y_{i_0}^\beta(0)$, the constant $\hbar > 0$, and the neighborhood $\mathcal{U} \subset \mathcal{P} = \mathcal{P}^{3/2}$ as in 5.6.4.

Proof of Theorem 5.4.5. The proof is almost verbatim the same as that of Theorem 5.4.3. In Step 1 we must construct the required map

$$(T_0, \infty) \rightarrow \mathcal{M}^0(x^\alpha, y^\beta; \{j_\lambda, H_\lambda, J_\lambda\}_\lambda) : T \mapsto (\lambda_T, u_T).$$

The only difference to the proof of Step 1 in Theorem 5.4.3 is that \mathcal{M}^- is now a set of pairs (λ, w^-) , where $\lambda \in [0, 1]$ and $w^- \in W^{2,2}(\Sigma_0, M)$ satisfies the same conditions as before with (j, H, J) replaced by $(j_\lambda, H_\lambda, J_\lambda)$. The map $\iota^- : \mathcal{M}^- \rightarrow \mathcal{P}$ is then given by

$$\iota^-(\lambda, w^-) := w^-(\iota_{i_0}^\beta(s_0, \cdot)).$$

The remainder of the proof of Step 1 is verbatim the same as in the proof of Theorem 5.4.3. The proof that the resulting map $T \mapsto (\lambda_T, u_T)$ satisfies the assertions (i), (ii), (iii), and (iv) of Theorem 5.4.5 can also be carried over word by word from the proof of Theorem 5.4.3. The details can be safely left to the reader. This proves Theorem 5.4.5. \square

Catenation

In this section we prove Theorem 5.4.8.

5.6.6. Let $(\Sigma^{\alpha\beta}, L^{\alpha\beta})$, $(\Sigma^{\beta\gamma}, L^{\beta\gamma})$, and $(j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$, $(j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$ be as in 5.4.6, and let

$$u^{\alpha\beta} \in \mathcal{M}^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta}), \quad u^{\beta\gamma} \in \mathcal{M}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$$

and $W^{\alpha\beta} \subset \Sigma^{\alpha\beta} \setminus \text{im } \iota^{\beta,+}$, $W^{\beta\gamma} \subset \Sigma^{\beta\gamma} \setminus \text{im } \iota^{\beta,-}$ be as in the assumptions of Theorem 5.4.8. Let $\mathcal{H}^{\alpha\beta}$ and $\mathcal{H}^{\beta\gamma}$ be as in 5.4.7.

5.6.7. For $i \in I^\beta$ let

$$\mathcal{P}_i := \mathcal{P}^{3/2}(H_i^\beta, J_i^\beta)$$

be the path space associated to the pair (H_i^β, J_i^β) as in 4.2.9. The proof of Theorem 5.4.8 involves the product space

$$\mathcal{P} := \prod_{i \in I^\beta} \mathcal{P}_i$$

Thus the elements of \mathcal{P} are tuples $\gamma = \{\gamma_i\}_{i \in I^\beta}$ with $\gamma_i \in \mathcal{P}_i$.

For each $i \in I^\beta$ choose open neighborhoods $U_i, V_i \subset M$ of $x_i^\beta(0)$ as in 4.1.6, choose a constant $\hbar > 0$ such that the assertion of Theorem 2.1.4 holds for each i with $U = U_i, V = V_i, \Lambda = \{x_i^\beta(0)\}$, and choose neighborhoods $\mathcal{U}_i \subset \mathcal{P}_i$ of x_i^β and a constant $T_0 > 0$ such that the assertions of Theorem 4.1.8 are satisfied with $\mathcal{U} = \mathcal{U}_i$ and $x = x_i^\beta$. The Hilbert manifolds $\mathcal{M}^\infty(x_i^\beta, \mathcal{U}_i)$ and $\mathcal{M}^T(x_i^\beta, \mathcal{U}_i)$ are defined by (4.16) and the embeddings

$$\iota_i^\infty : \mathcal{M}^\infty(x_i^\beta, \mathcal{U}_i) \rightarrow \mathcal{P}_i \times \mathcal{P}_i, \quad \iota_i^T : \mathcal{M}^T(x_i^\beta, \mathcal{U}_i) \rightarrow \mathcal{P}_i \times \mathcal{P}_i$$

for $T \geq T_0$ and $i \in I^\beta$ are defined by (4.13). Denote their images by

$$\mathcal{W}_i^\infty := \iota_i^\infty(\mathcal{M}^\infty(x_i^\beta, \mathcal{U}_i)), \quad \mathcal{W}_i^T := \iota_i^T(\mathcal{M}^T(x_i^\beta, \mathcal{U}_i)).$$

The products

$$\mathcal{W}^\infty := \prod_{i \in I^\beta} \mathcal{W}_i^\infty, \quad \mathcal{W}^T := \prod_{i \in I^\beta} \mathcal{W}_i^T$$

are submanifolds of $\mathcal{P} \times \mathcal{P}$ (of infinite dimension and infinite codimension). By Theorem 4.1.8 the submanifolds \mathcal{W}^T converge to \mathcal{W}^∞ in the C^1 topology as T tends to infinity.

Proof of Theorem 5.4.8. The proof has three steps.

Step 1. *Construction of the solutions*

$$\begin{aligned} u_h^{\alpha\beta} &\in \mathcal{M}^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta} + h^{\alpha\beta}, J^{\alpha\beta}), \\ u_h^{\beta\gamma} &\in \mathcal{M}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma} + h^{\beta\gamma}, J^{\beta\gamma}), \\ u_{h,T} &\in \mathcal{M}^0(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H + h)_T^{\alpha\gamma}, J_T^{\alpha\gamma}) \end{aligned}$$

of the Floer equation for $h \in \mathcal{H}_0 \subset \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$ and $T \geq T_0$.

Choose $s_0 > 0$ so large that

$$\begin{aligned} \int_{s_0}^\infty \int_0^1 \left| \partial_s u^{\alpha\beta}(\iota_i^{\beta,+}(s, t)) \right|_t^2 dt ds &< \hbar/2, \\ \int_{-\infty}^{-s_0} \int_0^1 \left| \partial_s u^{\beta\gamma}(\iota_i^{\beta,-}(s, t)) \right|_t^2 dt ds &< \hbar/2, \\ u^{\alpha\beta}(\iota_i^{\beta,+}(s_0, \cdot)) &\in \mathcal{U}_i, \quad u_i^{\beta\gamma}((\iota_i^{\beta,-}(-s_0, \cdot))) \in \mathcal{U}_i, \quad i \in I^\beta. \end{aligned} \tag{5.61}$$

Then

$$(u^{\alpha\beta}(\iota_i^{\beta,+}(s_0, \cdot)), u_i^{\beta\gamma}(\iota_i^{\beta,-}(-s_0, \cdot))) \in \mathcal{M}^\infty(x_i^\beta, \mathcal{U}_i), \quad i \in I^\beta.$$

Define

$$\begin{aligned} \Sigma_0^{\alpha\beta} &:= \Sigma^{\alpha\beta} \setminus \bigcup_{i \in I^\beta} \iota_i^{\beta,+}((s_0, \infty) \times [0, 1]), \\ \Sigma_0^{\beta\gamma} &:= \Sigma^{\beta\gamma} \setminus \bigcup_{i \in I^\beta} \iota_i^{\beta,-}((-\infty, -s_0) \times [0, 1]). \end{aligned}$$

(See equation (5.21).) For $h = (h^{\alpha\beta}, h^{\beta\gamma}) \in \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$ define

$$\begin{aligned} \mathcal{M}_h^{\alpha\beta} &:= \left\{ w^{\alpha\beta} \in W_{\text{loc}}^{2,2}(\Sigma_0^{\alpha\beta}, M) \left| \begin{array}{l} w^{\alpha\beta} \text{ satisfies (5.23) for } H^{\alpha\beta} + h^{\alpha\beta}, \\ E_H(w^{\alpha\beta}) < \infty, \\ \lim_{s \rightarrow -\infty} w^{\alpha\beta}(\iota_i^{\alpha,-}(s, t)) = x_i^\alpha(t), i \in I^\alpha \end{array} \right. \right\}, \\ \mathcal{M}_h^{\beta\gamma} &:= \left\{ w^{\beta\gamma} \in W_{\text{loc}}^{2,2}(\Sigma_0^{\beta\gamma}, M) \left| \begin{array}{l} w^{\beta\gamma} \text{ satisfies (5.23) for } H^{\beta\gamma} + h^{\beta\gamma}, \\ E_H(w^{\beta\gamma}) < \infty, \\ \lim_{s \rightarrow \infty} w^{\beta\gamma}(\iota_i^{\gamma,+}(s, t)) = x_i^\gamma(t), i \in I^\gamma \end{array} \right. \right\}. \end{aligned}$$

These spaces are Hilbert manifolds. The restriction maps

$$\iota_h^{\alpha\beta} : \mathcal{M}_h^{\alpha\beta} \rightarrow \mathcal{P}, \quad \iota_h^{\beta\gamma} : \mathcal{M}_h^{\beta\gamma} \rightarrow \mathcal{P},$$

defined by

$$\iota_h^{\alpha\beta}(w^{\alpha\beta}) := \left\{ w^{\alpha\beta}(\iota_i^{\beta,+}(s_0, \cdot)) \right\}_{i \in I^\beta}, \quad \iota_h^{\beta\gamma}(w^{\beta\gamma}) := \left\{ w^{\beta\gamma}(\iota_i^{\beta,-}(-s_0, \cdot)) \right\}_{i \in I^\beta}$$

are injective immersions (see section 5.5).

By our transversality hypothesis, the map

$$\iota_0^{\alpha\beta} \times \iota_0^{\beta\gamma} : \mathcal{M}_0^{\alpha\beta} \times \mathcal{M}_0^{\beta\gamma} \rightarrow \mathcal{P} \times \mathcal{P} \quad (5.62)$$

(associated to $h = 0$) intersects the submanifold \mathcal{W}^∞ transversally in the point

$$\left\{ (u^{\alpha\beta}(\iota_i^{\beta,+}(s_0, \cdot)), u^{\beta\gamma}(\iota_i^{\beta,-}(-s_0, \cdot))) \right\}_{i \in I^\beta} = \left(\iota_0^{\alpha\beta}(u^{\alpha\beta}|_{\Sigma_0^{\alpha\beta}}), \iota_0^{\beta\gamma}(u^{\beta\gamma}|_{\Sigma_0^{\beta\gamma}}) \right).$$

Moreover, the Fredholm index is zero, so the intersection point is isolated. Hence, by Theorem 4.1.8 and the implicit function theorem, there exists a

number $T_1 > 0$ and a convex neighborhood $\mathcal{H}_0 \subset \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$ of zero (open in the C^2 -topology) such that, for every $T \geq T_1$ and every $h \in \mathcal{H}_0$, the map

$$\iota_h^{\alpha\beta} \times \iota_h^{\beta\gamma} : \mathcal{M}_h^{\alpha\beta} \times \mathcal{M}_h^{\beta\gamma} \rightarrow \mathcal{P} \times \mathcal{P} \quad (5.63)$$

is transverse to \mathcal{W}^∞ and \mathcal{W}^T in a neighborhood of the pair

$$(u^{\alpha\beta}|_{\Sigma_0^{\alpha\beta}}, u^{\beta\gamma}|_{\Sigma_0^{\beta\gamma}}) \in \mathcal{M}_0^{\alpha\beta} \times \mathcal{M}_0^{\beta\gamma} \quad (5.64)$$

and has a unique intersection point with each of these submanifolds in that neighborhood. In other words, the following holds.

(I) For every $h \in \mathcal{H}_0$ there exists a unique pair $(u_h^{\alpha\beta}, u_h^{\beta\gamma}) \in \mathcal{M}_h^{\alpha\beta} \times \mathcal{M}_h^{\beta\gamma}$ near the pair (5.64) such that

$$(\iota_h^{\alpha\beta}(u_h^{\alpha\beta}), \iota_h^{\beta\gamma}(u_h^{\beta\gamma})) \in \mathcal{W}^\infty.$$

Thus there is a unique tuple $(w_{i,h}^+, w_{i,h}^-) \in \mathcal{M}^\infty(x_i^\beta, \mathcal{U}_i)$, $i \in I^\beta$, such that, for every $t \in [0, 1]$ and every $i \in I^\beta$,

$$w_{i,h}^+(0, t) = u_h^{\alpha\beta}(\iota_i^{\beta,+}(s_0, t)), \quad w_{i,h}^-(0, t) = u_h^{\beta\gamma}(\iota_i^{\beta,-}(-s_0, t)).$$

(II) For every $h \in \mathcal{H}_0$ and every $T \geq T_1$ there exists a unique pair

$$(u_{h,T}^{\alpha\beta}, u_{h,T}^{\beta\gamma}) \in \mathcal{M}_h^{\alpha\beta} \times \mathcal{M}_h^{\beta\gamma}$$

near the pair (5.64) such that

$$(\iota_h^{\alpha\beta}(u_{h,T}^{\alpha\beta}), \iota_h^{\beta\gamma}(u_{h,T}^{\beta\gamma})) \in \mathcal{W}^T.$$

Thus there is a unique tuple $w_{i,h,T} \in \mathcal{M}^T(x_i^\beta, \mathcal{U}_i)$, $i \in I^\beta$, such that, for every $t \in [0, 1]$ and every $i \in I^\beta$,

$$w_{i,h,T}(-T, t) = u_{h,T}^{\alpha\beta}(\iota_i^{\beta,+}(s_0, t)), \quad w_{i,h,T}(T, t) = u_{h,T}^{\beta\gamma}(\iota_i^{\beta,-}(-s_0, t)).$$

For $h \in \mathcal{H}_0$ define the maps $u_h^{\alpha\beta} : \Sigma^{\alpha\beta} \rightarrow M$ and $u_h^{\beta\gamma} : \Sigma^{\beta\gamma} \rightarrow M$ by

$$\begin{aligned} u_h^{\alpha\beta}(z) &:= \begin{cases} u_h^{\alpha\beta}(z), & \text{if } z \in \Sigma_0^{\alpha\beta}, \\ w_{i,h}^+(s, t), & \text{if } z = \iota_i^{\beta,+}(s + s_0, t), i \in I^\beta, \end{cases} \\ u_h^{\beta\gamma}(z) &:= \begin{cases} u_h^{\beta\gamma}(z), & \text{if } z \in \Sigma_0^{\beta\gamma}, \\ w_{i,h}^-(s, t), & \text{if } z = \iota_i^{\beta,-}(s - s_0, t), i \in I^\beta. \end{cases} \end{aligned} \quad (5.65)$$

For $h \in \mathcal{H}_0$ and $T \geq T_0 := s_0 + T_1$ define the map $u_{h,T}^{\alpha\gamma} : \Sigma_T^{\alpha\gamma} \rightarrow M$ by

$$u_{h,T}^{\alpha\gamma}(z) := \begin{cases} u_{h,T-s_0}^{\alpha\beta}(z), & \text{if } z \in \Sigma_0^{\alpha\beta}, \\ u_{h,T-s_0}^{\beta\gamma}(z), & \text{if } z \in \Sigma_0^{\beta\gamma}, \\ w_{i,h,T-s_0}(s,t), & \text{if } z = \iota_i^{\beta,+}(s+T,t) \cong \iota^{\beta,-}(s-T,t) \\ & \text{and } |s| \leq T-s_0, i \in I^\beta. \end{cases} \quad (5.66)$$

Then

$$\begin{aligned} u_h^{\alpha\beta} &\in \mathcal{M}^0(x^\alpha, x^\beta; j^{\alpha\beta}, H^{\alpha\beta} + h^{\alpha\beta}, J^{\alpha\beta}), \\ u_h^{\beta\gamma} &\in \mathcal{M}^0(x^\beta, x^\gamma; j^{\beta\gamma}, H^{\beta\gamma} + h^{\beta\gamma}, J^{\beta\gamma}), \\ u_{h,T} &\in \mathcal{M}^0(x^\alpha, x^\gamma; j_T^{\alpha\gamma}, (H+h)_T^{\alpha\gamma}, J_T^{\alpha\gamma}) \end{aligned}$$

for every $h \in \mathcal{H}_0$ and every $T \geq T_0$. This proves Step 1.

Step 2. *The maps $u_{h,T}$ constructed in Step 1 satisfy conditions (i), (ii), and (iii) in Theorem 5.4.8.*

That the solutions constructed in Step 1 are regular (i.e. the linearized operators are bijective) follows directly from transversality in the path space $\mathcal{P} \times \mathcal{P}$. This shows that the functions $u_h^{\alpha\gamma}$ and $u_{h,T}^{\alpha\gamma}$ satisfy condition (i) in Theorem 5.4.8. The convergence statement in (ii) follows immediately from the construction. The action identity in (iii) follows from the fact that the catenation of $u^{\alpha\beta}$ and $u^{\beta\gamma}$ is homotopic to $u_{h,T}^{\alpha\gamma}$ for all h and T . This proves Step 2.

Step 3. *The maps $u_{h,T}$ constructed in Step 1 satisfy condition (iv) in Theorem 5.4.8, after shrinking \mathcal{H}_0 and increasing T_0 , if necessary, and for $\delta_0 > 0$ sufficiently small.*

Suppose, by contradiction, that the solutions $u_{h,T}^{\alpha\gamma}$ of the Floer equation, constructed in Step 1, do not satisfy assertion (iv) in Theorem 5.4.8 for any triple $(\delta_0, \mathcal{H}_0, T_0)$. Then there exists a sequence of perturbations

$$h_i = (h_i^{\alpha\beta}, h_i^{\beta\gamma}) \in \mathcal{H}^{\alpha\beta} \times \mathcal{H}^{\beta\gamma}$$

converging to zero in the C^2 topology, a sequence $T_i \rightarrow \infty$, and a sequence

$$u_i \in \mathcal{M}^0(x^\alpha, x^\gamma; j_{T_i}^{\alpha\gamma}, (H+h_i)_{T_i}^{\alpha\gamma}, J_{T_i}^{\alpha\gamma})$$

such that, for every $i \in \mathbb{N}$, the following holds.

- (a) $u_i \neq u_{h_i, T_i}^{\alpha\gamma}$.
- (b) $\mathcal{A}_H(u_i) = \mathcal{A}_H(u^{\alpha\beta}) + \mathcal{A}_H(u^{\beta\gamma})$.
- (c) $\lim_{i \rightarrow \infty} \sup_{W^{\alpha\beta}} d(u_i, u_h^{\alpha\beta}) = 0$ and $\lim_{i \rightarrow \infty} \sup_{W^{\beta\gamma}} d(u_i, u_h^{\beta\gamma}) = 0$.

By the standard elliptic bootstrapping, bubbling, and removal of singularities argument, we may assume as before, after passing to a subsequence if necessary, that the restriction of u_i to $\Sigma^{\alpha\beta}$, respectively $\Sigma^{\beta\gamma}$, converges in $W_{\text{loc}}^{2,2}$ to a smooth finite energy solution $v^{\alpha\beta}$, respectively $v^{\beta\gamma}$, of the Floer equation for $(j^{\alpha\beta}, H^{\alpha\beta}, J^{\alpha\beta})$, respectively $(j^{\beta\gamma}, H^{\beta\gamma}, J^{\beta\gamma})$. By (c) and unique continuation, we must have $v^{\alpha\beta} = u^{\alpha\beta}$ and $v^{\beta\gamma} = u^{\beta\gamma}$. By (b) there is no loss of energy in the limit and hence no bubbling or splitting off of Floer trajectories can occur. Hence the sequence $(h_i, u_i|_{\Sigma_0^{\alpha\beta}})$ converges in the topology of the fiber bundle

$$\mathcal{M}^{\alpha\beta} := \bigcup_{h \in \mathcal{H}_0} \{h\} \times \mathcal{M}_h^{\alpha\beta} \rightarrow \mathcal{H}_0$$

to the pair $(0, u^{\alpha\beta}|_{\Sigma_0^{\alpha\beta}})$. Likewise, the sequence $(h_i, u_i|_{\Sigma_0^{\beta\gamma}})$ converges in the topology of $\mathcal{M}^{\beta\gamma} := \bigcup_{h \in \mathcal{H}_0} \{h\} \times \mathcal{M}_h^{\beta\gamma}$ to the pair $(0, u^{\beta\gamma}|_{\Sigma_0^{\beta\gamma}})$. Hence it follows from the construction in Step 1 and the implicit function theorem in the path space $\mathcal{P} \times \mathcal{P}$ that $u_i = u_{h_i, T_i}^{\alpha\gamma}$ for i sufficiently large, in contradiction to (a). This contradiction proves Step 3 and Theorem 5.4.8. \square

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